A Homework: Origins (due Thu 20th Feb 5pm)

A.1 Particle physics units

Units: $1\text{ eV} = 1.6 \times 10^{-19}$ J, $c = 3 \times 10^{8}$ m s^{-1}, $\hbar c = 197$ MeV fm, $\hbar = 1.05 \times 10^{-34}$ J s.

(a) Convert the electron and proton mass is $m_e = 9.11 \times 10^{-31}$ kg and $m_p = 1.67 \times 10^{-27}$ kg into units of MeV. Numerically estimate the reduced Compton wavelengths $\lambda\!\!\!^{-}_C = \hbar/mc$ of an electron, proton, and a Higgs boson $m_h = 125$ GeV.

(b) Muons undergo beta decay at a rate given by $\Gamma = (G_F^2 m_\mu^5)/(192\pi^3)$. Given the Fermi constant is $G_F = 1.17 \times 10^{-5}$ GeV^{-2} and muon mass $m_\mu = 106$ MeV, numerically calculate the rest lifetime of the muon $\tau = \hbar/\Gamma$ in seconds.

(c) From the Lorentz force $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B}$ and centripetal acceleration $a = v^2/R$, derive

$$\left(\frac{p}{\text{GeV c}^{-1}}\right) = k\left(\frac{B}{1\text{ T}}\right)\left(\frac{R}{1\text{ m}}\right). \tag{A.1}$$

relating a particle with unit charge $q = 1.6 \times 10^{-19}$ C, momentum $p = \gamma m v$ at constant velocity perpendicular to the magnetic field B, bending radius R, and $k \approx 0.3$. Note that the SI units for Tesla are $[B] = [f]/[qv] = \text{N}/(\text{C m s}^{-1})$.

A.2 Cosmic-ray relativistic kinematics

(a) In the lab frame, a cosmic ray with four-momentum $\mathsf{P}_C^\mu = (E, \mathbf{p})^{\mathrm{T}}$ strikes a stationary nucleus with mass m_N. Show that the centre-of-mass energy is $E_{\text{CM}} \approx \sqrt{2m_N E}$ and write the condition this holds. Use this to estimate the proton energy needed to strike a stationary oxygen nucleus ($m_{\text{oxygen}} \approx 16 m_{\text{proton}}$) with LHC energies: $E_{\text{CM}} = 7$ and 13.6 TeV.

(b) In the "GZK process", a proton strikes a primordial cosmic photon $p + \gamma_{\text{cosmic}} \to n + \pi^+$, producing a neutron and pion at threshold. Apply energy–momentum conservation to show the minimum proton energy is $E_p \approx [(m_n + m_\pi)^2 - m_p^2]/4E_\gamma$, and estimate its value given $E_\gamma = 0.23$ meV, $m_p = 938.3$ MeV, $m_n = 939.6$ MeV, $m_\pi = 139.6$ MeV.

A.3 Dirac equation

(a) From the Weyl representation of γ^μ matrices eq. (3.50), write $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ as 4×4 matrices. For $\{\gamma^0, \gamma^1\}$ and $\{\gamma^1, \gamma^1\}$, show by explicit multiplication they satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

(b) The Dirac equation in its coupled Weyl form is $\mathrm{i}\sigma^\mu \partial_\mu \chi = m\phi$ and $\mathrm{i}\bar{\sigma}^\mu \partial_\mu \phi = m\chi$. Using the Pauli matrices (section 3.3) and definition of four-derivatives, write these as 2×2 matrix equations in terms of energy $E = \mathrm{i}\partial_t$ and momentum $\mathbf{p} = -\mathrm{i}\boldsymbol{\nabla}$ operators.