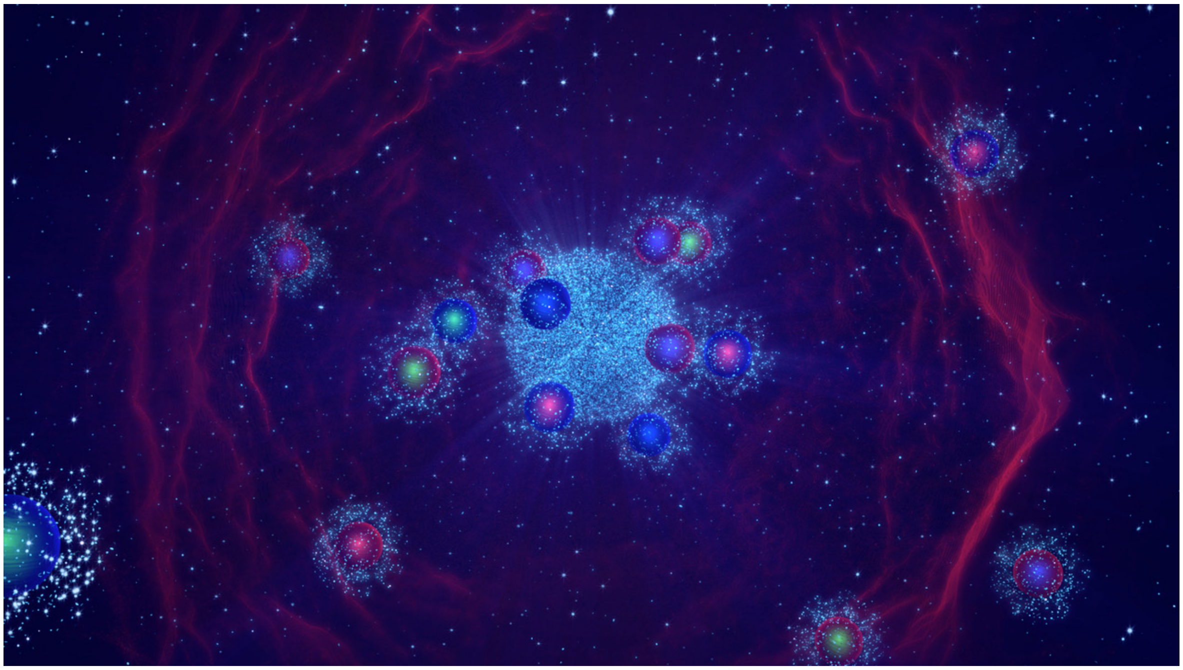


Particle Physics

New York Lectures

Jesse Liu

Department of Physics, New York University,
726 Broadway, New York, NY 10003, USA

These lecture notes accompany the NYU graduate introduction to particle physics (PHYS-GA 2027) in Spring 2025. This course presents the phenomenological and experimental discoveries leading to the Standard Model. After introducing the historical foundations, the class surveys the empirical evidence underpinning quantum electrodynamics, the strong force, and electroweak interactions. The level is for graduate students exposed to special relativity, quantum mechanics, and electromagnetism.

0 Syllabus

Course: NYU PHYS-GA 2027 Particle Physics
Semester: Spring 2025
Instructor: Jesse Liu
Lectures: Tuesday and Thursdays, 11:00–12:15, 726 Broadway, Room 1025
Office hours: Wednesdays, 11:00–12:00, Room 852 or by appointment

0.1 Course overview

This is the New York University graduate introduction to the Standard Model of particle physics (PHYS-GA 2027). The semester comprises 14 weeks of classes with two 75-minute lectures per week, totalling 27 planned lectures slots. Like previous versions of this class, the finals week is devoted to student presentations. Below is a lecture plan, which is subject to change based on the pace of delivery (sections of lecture notes in parentheses):

Historical origins

1. Course overview, motivation and introduction to the Standard Model (1.1–1.3).
2. Radioactivity, evidence for neutrinos, ionisation and cloud chambers (2.1).
3. Proton and neutron, cosmic rays for positron and muon discoveries (2.2–2.3).
4. Relativity and quantum mechanics review, constructing Dirac equation (3.1–3.2).
5. Weyl equations, helicity and chirality, Dirac mass, antimatter (3.3).
6. Spinor rotation, non-relativistic limit to Pauli equation: gyromagnetic factor (3.4).

Quantum electrodynamics

7. Feynman diagrams, electromagnetic scattering, virtual particles, propagators (4.1–4.3).
8. Gauge theory of forces, local symmetry, gauge fixing, Feynman rules (4.4).
9. Accelerators, cross-sections, Fermi's golden rule, density of states (5.1–5.4).
10. Electron–positron annihilation, spinor–helicity analysis, resonances (6.1–6.3).
11. Loop effects: Lamb shift, anomalous magnetic moment, running coupling (7.1–7.3).

Strong force

12. Particle zoo: nuclear moments, emulsions, cyclotrons, bubble chambers (8.1–8.4).
13. Quark model, strangeness, meson and baryon multiplets, colour charge (9.1–9.4).
14. Nuclear form factors, deep inelastic scattering and evidence for partons (10.1–10.2).
15. Evidence for quarks and their quantum properties, charmonium J/ψ (10.3–10.4).

16. Yang–Mills theory, evidence for colour, gluons, asymptotic freedom (11.1–11.4).

Electroweak interactions

17. Low-energy beta decay: Fermi theory, neutrino detection, parity violation (12–12.3).
18. Helicity suppression, flavour mixing, charge-parity violation in kaons (12.3–12.5).
19. Collider experiments, particle–matter interactions, terascale detectors (13.1–13.3).
20. Collider kinematics and event reconstruction, particle identification (13.4–13.5).
21. Abelian Higgs model, Glashow–Salam–Weinberg model (14.1–14.2).
22. Generating gauge boson masses, electroweak unification (14.3).
23. W and Z boson discovery, evidence for three light neutrinos (14.4–14.5).
24. Higgs boson discovery, ATLAS and CMS measurements (15.1–15.2).
25. Higgs–Yukawa couplings, Cabbibo–Kobayashi–Maskawa matrix (15.3–15.4).
26. Evidence for massive neutrinos, solar and atmospheric oscillations (16.1–16.4).

Outlook

27. Open questions, motivation for physics beyond the Standard Model.

Structure and approach

This graduate class has been taught by many esteemed colleagues at NYU with various levels and approaches. In recent years, the class required Quantum Field Theory I (PHYS-GA 2058), but this version does not. I instead return to more conventional introductory particle physics textbooks for advanced undergraduates and beginning graduates. This course develops relativistic quantum mechanics and emphasises experimental discoveries, assuming a standard US undergraduate physics curriculum as pre-requisites. This enables the course to complement rather than overlap too much with the theoretical QFT classes.

Given the breadth of particle physics, the choice of topics can neither be an exhaustive nor encyclopedic list of all particles, processes, calculations, and experiments. The aim instead is to convey the foundations and build intuition using illustrative examples as a springboard for more specialised study. This includes the literature review presentations in this class and the start of your graduate careers.

The course is organised into four parts covering how the Standard Model was discovered: (i) historical origins discussing motivation alongside foundational discoveries, (ii) quantum electrodynamics as the prototypical gauge theory, (iii) strong force showing how quarks and gluons emerged from the particle zoo, (iv) electroweak interactions from parity violation in low-energy beta decay to the Higgs boson and neutrino oscillations. The ordering may appear superficially historical, but worry not, the structure largely charts increasing energy scales and decreasing interaction strengths. This unsurprisingly coincides with technological advances, with more powerful machines simply taking longer to develop.

The lectures endeavour to weave selected historical, phenomenological, and experimental perspectives that make particle physics such a fascinating subject. Indeed understanding the *process* of discovery is as interesting as the discovery itself, especially for aspiring researchers learning how to uncover new knowledge themselves. I anticipate a mix of board work for mathematical derivations supplemented by slides to show more detailed figures and data. As time permits, the last lecture(s) may introduce the motivation for physics beyond the Standard Model and/or special research topics.

Pre-requisites

Mathematics: linear algebra (matrix multiplication, eigenvalues), complex numbers, vector calculus (integration in spherical coordinates). Physics: special relativity (space-time metric, Lorentz boosts), electromagnetism (Maxwell's equations, electromagnetic waves), quantum mechanics (Schrödinger equation, Pauli matrices, perturbation theory).

Recommended but not required: Quantum Field Theory I (PHYS-GA 2058) and II (PHYS-GA 2077) covers similar topics with more theoretical emphasis.

Assessment

- Grade: 60%. Homework. There are 5 problem sets planned based on lecture content.
- Grade: 40%. Research review. 10–15 minute presentation with slides. Attendance is required. Usual class location and time during finals week.

Presentation topics

In lieu of a final exam, each student selects a historical discovery or ongoing experiment related to particle physics and prepares a 10–15 minute talk; I recommend 10 slides of content. The presentation comprises a research literature review using slides during the regular class times of finals week. The topic is mutually agreed upon with me in advance (to ensure breadth by not too many people choosing the same topic). Suggested topics include:

1. Discovery of tau-lepton and/or tau-neutrino.
2. Discovery of top quark at Tevatron, Fermilab.
3. Higgs potential shape and Higgs self-coupling probes at LHC.
4. Tetraquarks and pentaquarks at LHCb Experiment.

5. Charge-parity violation in B-mesons.
6. Laboratory neutrino oscillations and charge-parity violation e.g. MicroBOONE, DUNE.
7. Majorana vs. Dirac neutrinos and neutrinoless double beta decay e.g. SNO+.
8. Measuring the muon anomalous magnetic moment ($g-2$) at Fermilab.
9. Physics of High-Luminosity LHC and detector upgrades.
10. Future high-energy colliders (e.g. FCC, Muon Collider) and their physics goals.
11. Searches for weak-scale dark matter at colliders and direct detection experiments.
12. Searches for axion dark matter and axion-like particles.
13. Cosmic-ray observatories in space and terrestrial e.g. AMS, IceCube, Auger.
14. Accelerators and detectors in medical physics for imaging and radiotherapy.
15. Cosmic-ray muography for imaging volcanoes and archaeology sites.

0.2 Literature

There is no required textbook and I encourage you to browse your library to find your preferred text. This list also represents a bibliography I consulted (or studied as a student) in preparing the lecture notes.

Open access books. It is a truth universally acknowledged that textbooks are expensive. So I was happy to see certain texts recently made open access funded by the SCOAP^3 initiative coordinated by CERN, which is worth supporting:

- Giles Barr, Robin Devenish, Roman Walczak, Tony Weidberg, *Particle Physics in the LHC era* [1] (OUP 2016, Open Access Library[1]). Based on Oxford Part C master's level course; I myself took the graduate lectures in particle physics as a PhD student.
- Alessandro Bettini, *Introduction to Elementary Particle Physics* (3rd Edition, CUP 2024, open access on Cambridge Core[2]. This recently-updated textbook has particularly lucid accounts of the history behind experimental discoveries of particle physics.

Online lecture materials. A quick Internet search of "particle physics lectures" reveals many high-quality materials online. I list some from my academic heritage:

- **Advanced undergraduate level**
  Tina Potter teaches the Cambridge *Part II Nuclear and Particle Physics*[3] course for

[1] https://library.oapen.org/handle/20.500.12657/59108
[2] https://doi.org/10.1017/9781009440745
[3] https://www.hep.phy.cam.ac.uk/~chpotter/particleandnuclearphysics/mainpage.html

final (third) year undergraduates, which also covers nuclear physics. This is a standard introductory course for third-year British undergraduates, where I myself took the analogous lectures at Oxford taught by Alan Barr in 2014. I still possess some notes from my studies that I recycle here.

- **Graduate level**
  Chris Lester has over 600 slides for the Cambridge Natural Sciences *Part III Particle Physics*[4] master's level course with an experimental emphasis and complements the Thomson textbook. Theory counterparts are written up by David Tong[5], alongside Fernando Quevedo and Andreas Schachner[6].

  My education followed the *Standard Model* course at the Perimeter Institute, taught by Stefania Gori and Gordan Krnjaic with Daniel Wohns and Gang Xu in 2015; video recordings remain online[7] and the most recent 2025 edition is lectured by Sydek Ipek[8]. I also took an advanced PSI classes by David Morrissey who has clear SM notes[9] and Brian Shuve who has SM lectures recorded[10].

Canonical textbooks. Here are a few much-loved particle physics textbooks that do not assume quantum field theory, where the publication year is suggestive of whether post-LHC updates are included:

- David Griffiths, *Introduction to Elementary Particles* (2nd Edition, Wiley 2004). An old favourite with good coverage of history and Feynman rule calculations.
- Andrew J. Larkoski. *Elementary Particle Physics: An Intuitive Introduction* (CUP 2019). This recent textbook provides pedagogical coverage of LHC-era analysis topics.
- Brian R. Martin and Graham Shaw, *Particle Physics* (4th Edition, Manchester Physics Series 2017). Discussions on the quark model and interactions are especially clear, with good introductory chapters about accelerator and detector techniques.
- Donald Perkins, *Introduction to High Energy Physics* (4th Edition, CUP 2000). Old but classic with good balance of experiment and phenomenology, which trained many generations of professionals before the Higgs discovery.

[4] https://www.hep.phy.cam.ac.uk/~lester/teaching/partIIIparticles/welcome.html
[5] https://www.damtp.cam.ac.uk/user/tong/standardmodel.html
[6] https://arxiv.org/abs/2409.09211
[7] https://pirsa.org/c15001
[8] https://pirsa.org/c25003
[9] https://particletheory.triumf.ca/PHYS528/
[10] https://brianshuvephysics.com/materials

- Mark Thomson, *Modern Particle Physics* (CUP 2013)[11]. Modern classic written for the Cambridge Natural Sciences Part III course by the new CERN Director General.

Further reading. More specialised reading and summer school materials for the aspiring particle physics researcher:

- Particle Data Group, *Review of Particle Physics* (PRD 2024)[12]. *The* standard reference compiling up-to-date values and reviews, regularly updated by the group.
- Theoretical Advanced Study Institute (2024)[13]. Colorado summer school lectures providing advanced training for aspiring theorists and phenomenologists.
- Fermilab–CERN Hadron Collider Physics School (2024)[14]. The summer school alternates between Fermilab and CERN with slides available for advanced training aimed at aspiring collider physicists.
- Robert Cahn and Gerson Goldhaber, *The Experimental Foundations of Particle Physics* (CUP 2009). Nice account of the experimental evidence with prints of original discovery papers that led to the Standard Model.
- Howard Georgi, *Lie Algebras In Particle Physics: from Isospin To Unified Theories* (CRC Press 2000). For those who want to study the more mathematical aspects of group and representation theory underpinning particle physics.
- Glenn Knoll, *Radiation Detection and Measurement* (Wiley 2010). The standard reference for instrumentation underpinning detector physics.
- Tom Lancaster and Stephen Blundell, *Quantum Field Theory for the Gifted Amateur*[15] (OUP 2014). An accessible and lucid quantum field theory text written by condensed matter experimentalists that I found very helpful as a student.

[11] https://www.hep.phy.cam.ac.uk/~thomson/MPP/ModernParticlePhysics.html
[12] https://pdg.lbl.gov/, https://doi.org/10.1103/PhysRevD.110.030001
[13] https://sites.google.com/colorado.edu/tasi-2024-hub/home
[14] https://indico.fnal.gov/event/63696/
[15] https://academic.oup.com/book/36442

0.3 About these notes

For my own organisation, I have typeset lecture notes for this course, which grew from preparing for the class I first taught in Spring 2025. It comprises a synthesis of standard textbooks, journal articles, and online material with guided explanations for education purposes. I sketch numerous figures myself and endeavour to provide the sources for other figures in the captions.

These notes are intended to help you navigate the subject and certainly do not replace the wealth of excellent literature listed. Nonetheless this first version will inevitably have some rough edges. As a work-in-progress document created by a fallible human, feel free to send corrections to typographical errors. Presentational clarity and discussions will hopefully improve as future iterations of this class arise. Hopefully you may find them useful.

Miscellaneous

I typeset these notes using LaTeX adapting the JHEP template[16]. I draw various diagrams using `figma`, `feynmp-auto`, `tikz.net`. My typing hands and spellcheck dictionary are set to 'British English' e.g. aluminium, anti-clockwise, centre, colour, flavour, fulfil, labelled, metre, normalise, parametrise vs aluminum, counter-clockwise, center, color, flavor, fulfill, labeled, meter, normalize, parameterize. The title page image shows an artist's impression of Higgs field interactions from CERN[17]. These notes were initially prepared while generously supported by a Junior Research Fellowship at Trinity College, University of Cambridge.

Version history

0.1: preliminary draft 20th January 2025

[16] https://jhep.sissa.it/jhep/help/JHEP_TeXclass.jsp
[17] https://home.cern/science/physics/

Contents

I Historical origins

1 Introduction

Why study nature at the smallest scales? Why understand the fundamental forces of the universe? Why do we invest resources operating vast collider experiments? You are probably reading these notes because you think particle physics is interesting enough to enrol in a graduate class. So *you* likely need no convincing and can merrily skip this motivational introduction. But outside the classroom, your friends, family and acquaintances may ask why you picked this class or why the basic sciences are worth prioritising, especially when other pressing problems exist in our world. These are fair and timeless questions. So for completeness, let us summarise the canonical reasons to study particle physics.

1.1 Why study particle physics?

Particle physics aims to understand the fundamental building blocks of nature, simply put:

"What is the world made of?"

The idea that we can divide what we see around us into indivisible constituents dates back to ancient philosophies and cultures. By identifying these parts and how they interact, we can not only explain natural phenomena with deeper principles but also engineer solutions to problems in our society. Scientific discovery follows an uneven but familiar cadence:

Discovery phase		**Empirical structure**		**Predictive theory**
Observations surprise and even seem chaotic	$\to$	Systematic characterisation reveals unexplained patterns	$\to$	Simple principles dynamically generate rich phenomena

We will see this process play out throughout the Standard Model, such as the particle zoo to quark model to flavour physics. Chemistry is the first success story that betrays an uncanny resemblance to particle physics, so let us start there.

The principled structure of chemistry

In the early nineteenth century, scientists noticed that chemical reactions proceed with integer ratios of elements. Water combined two parts hydrogen with one part oxygen and ammonia

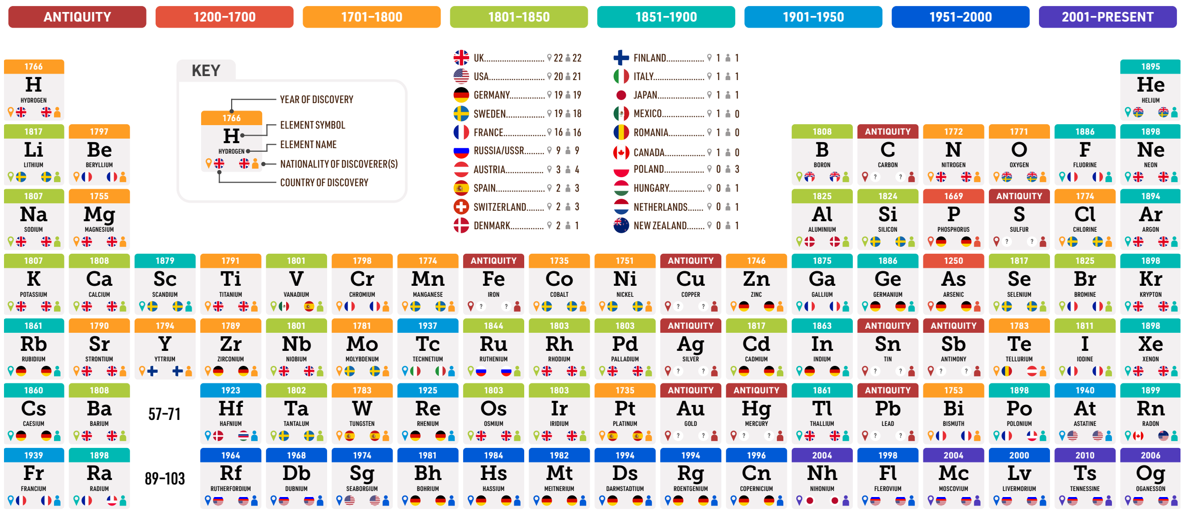


Figure 1: **Periodic table by discovery year**. This illustrates how empirical structure emerges much later than experimental discovery and detailed empirical characterisation. Image: Andy Brunning/Compound Interest (2019).

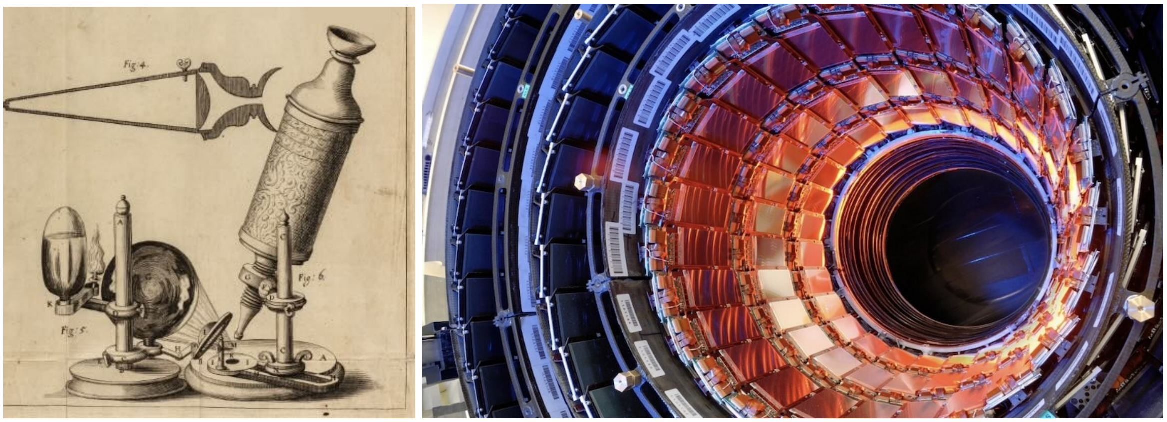
comprised three parts hydrogen to one part nitrogen. This led to John Dalton's "Law of Multiple Proportions" and 1807 proposal of atomic theory. The former appeared empirically successful but the reality of tiny unseeable atoms remained controversial for a century. By 1864, around fifty chemical elements were known with scientists such as John Newlands observing a mysterious eight-fold periodicity in a "Law of Octaves".

In 1869, Dmitri Mendeleev famously arranged the chemical elements into an organised table (figure 1). He used it to predict three new elements: gallium, germanium, and scandium. Elements are grouped by their empirical attributes: silvery solid alkalis and gaseous halogens react vigorously, while neighbouring noble gases stay inert. This pattern repeats with a mass periodicity of eight then eighteen for heavier elements. Structure was emerging after centuries of disjointed discoveries. But it was natural to ask why? How many more elements await discovery? Are deeper dynamics behind this structure?

In 1905, Albert Einstein applied statistical physics to Brownian motion to show atoms exist. Abridging the ensuing decades of revolutionary quantum mechanics and semesters of undergraduate physics to one sentence revealed just three subatomic building blocks:

$$\text{Atoms} : \{\, \text{protons} \,,\ \text{neutrons} \,,\ \text{electrons} \,\}\,. \tag{1.1}$$

From this mere handful of parts, we can build the hundreds of elements and their isotopes. How very elegant. Quantum mechanics and electromagnetism govern wavefunction orbitals



(a) Hooke's microscope c. 1665 (b) Compact Muon Solenoid (CMS) Experiment c. 2010

Figure 2: **Discovery instruments: microscopes four centuries apart**. Optical microscopes discovered biological cells in the seventeenth century, opening the field of microbiology. Today, detectors such as Compact Muon Solenoid at CERN comprise the most powerful microscopes probing 10^{-18} m. Images: Royal Society and CERN.

and ionisation energies, explaining why halogens are so reactive but not noble gases next door. This is the principled structure of chemistry. It endows the richness of molecular biology to material science and semiconductor electronics. It answers "how many more elements exist?": thankfully finite, just over hundred! Pack too many protons and neutrons into a nucleus, they become unstable and radioactively decay. This is a triumph for empirical reductionism. Just after the neutron discovery, the alluring "what is the world made of?" picture of (1.1) led Paul Dirac to reflect at the 1933 Solvay Conference:

> "If we consider protons and neutrons as elementary particles, we would have [with electrons] three kinds of elementary particles. . . This number may seem large but, from that point of view, two is already a large number."

We now know that this is far from the end of the story. It is the birth of particle physics.

Microscopes illuminate the microcosm

How do we know subatomic particles even exist? What instruments do we need to unveil the microcosm? Nature under microscopes is truly surprising (figure 2). Antonie van Leeuwenhoek and Robert Hooke peered through their microscopes bending light in the seventeenth century to unveil objects 100 times smaller than what the eye can resolve of around 0.1 mm. They revealed the building blocks of life: the existence of cells. Today, we overcome the diffraction limit of optical microscopes using wave-particle duality to probe ever smaller scales via higher energies. We build city-sized particle colliders and their cathedral-sized detectors as the most powerful microscopes probing 10^{-18} m length scales. Even with hindsight, it is unclear if or how pure philosophical or mathematical contemplation in a cave could have revealed the Standard Model without empirical guidance. Studying the experimental instruments revealing subatomic degrees of freedom is utterly worthwhile.

Stunning empirical verification

Why do we celebrate the Standard Model as a theory of nature? How well does it stand up to experimental scrutiny? Perturbation theory allows quantum field theory to predict observables that pass diverse experimental tests. The epitome of such empirical success is the electron gyromagnetic factor g_e from intrinsic spin. Its recent measurements and predictions agree to parts per trillion precision [2, 3]

$$g_e^{\text{meas}} = 2.002\,319\,304\,361, \tag{1.2}$$

$$g_e^{\text{pred}} = 2.002\,319\,304\,364. \tag{1.3}$$

This is among the most precisely tested quantities in nature. Few other empirical fields manage such feats. It is spectacular.

Yet a single observable alone is not what makes the Standard Model so successful. Its range of empirical validity extends to the highest laboratory energies. Electroweak theory and data predicted the existence of the top quark and Higgs boson. The Standard Model is a particular quantum field theory, and the fact it describes reality so well is why we trust its most peculiar predictions from anti-matter to vacuum polarisation.

Profound explanatory power

How does the Standard Model deepen our understanding of reality? What is its explanatory power? The Schrödinger equation cannot describe relativistic electrons or photons. Amazingly, making quantum mechanics consistent with special relativity explains many mysteries of undergraduate physics while revealing plenty more surprises. It tells us why the electron has spin half and its gyromagnetic factor is (nearly) two. We learn how two electrons can even be exactly identical. We find out nature gives us left and right handed electrons but we barely noticed until we realised the weak force cares. We upend our view of what inertial mass is: a scalar field gains a non-zero value to pair left with right handed electrons.

We reveal the uncertainty principle and mass–energy equivalence means the quantum vacuum is neither static nor empty. It is actually dynamical, a teeming sea of virtual particles and anti-particles popping in and out of existence. We learn that matter and charge are conserved due to symmetries in nature. While electromagnetism, the strong and weak nuclear forces could not behave more differently on first glance, we uncover that they actually share the same theoretical structure called gauge theories. These surprising but principled pictures of reality are radical departures from classical physics. It is why many find particle physics profound. Some might say beautiful.

Particle astrophysics and cosmology

How did the universe create all the matter we see? How do they propagate and interact in the cosmos? Particle physics is the study of our primordial origins (figure 3a). Big Bang cosmology (see the Cosmology course PHYS-GA 2052 for further details) implies the early universe saw far higher temperatures than today:

Electroweak transition: quarks & leptons gain mass	10^{-12} s	10^{15} K	100 GeV
Quark–hadron transition: quarks confine into protons	10^{-6} s	10^{12} K	100 MeV
Nucleosynthesis: protons & neutrons fuse into nuclei	3 mins	10^{9} K	100 keV

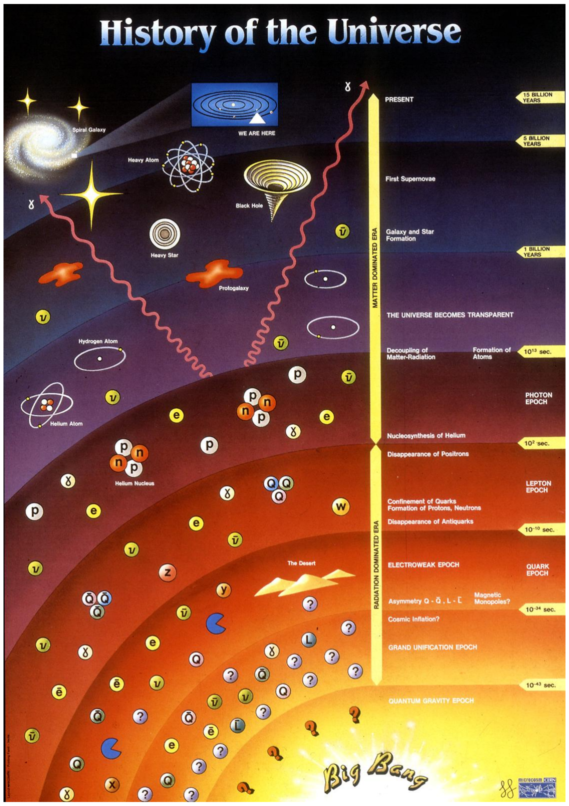
When the universe was about 10^{-12} s old, the temperature cooled to a balmy $k_B T \approx 100$ GeV, triggering electroweak symmetry breaking and giving gauge bosons, quarks and leptons finite mass. At 10^{-6} s, free quarks and gluons confine into protons and neutrons as the universe cools to $k_B T \approx 100$ MeV. Particle physics tells these phase transitions must have happened. For the first time in 13.8 billion years, we now have colliders that can recreate these extreme conditions of the early universe in the relative comforts of our laboratory.

Big Bang nucleosynthesis started fusing protons and neutrons in the first few minutes. It would take another 300 000 years before these nuclei bound to electrons to create atoms. The Standard Model particle content and interactions are now imprinted in the cosmic microwave background. This allows cosmology to measure the total Standard Model contribution (dominated by baryons) $\Omega_{\rm SM}$ to the energy budget of the universe [6]

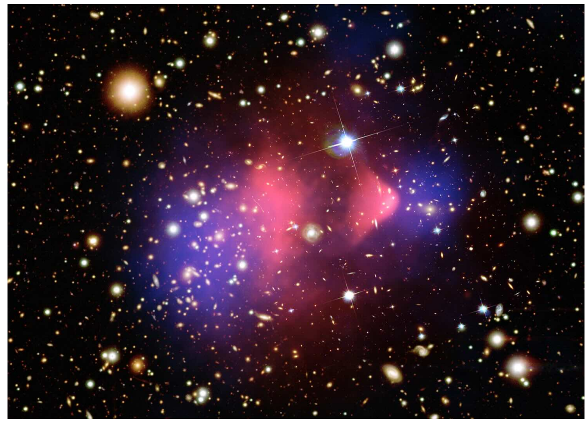
$$\Omega_{\rm SM} \simeq 0.049, \quad \Omega_{\rm DM} \simeq 0.26, \quad \Omega_{\rm DE} \simeq 0.69. \tag{1.4}$$

The remainder is enigmatically named dark matter $\Omega_{\rm DM}$ and dark energy $\Omega_{\rm DE}$, whose microscopic properties remain major research questions (figure 3b).

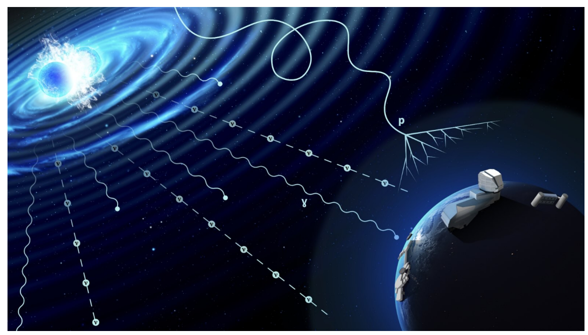
Astronomy historically relied on observing light using telescopes. These “messengers” now extend photons to other electrons, positrons, protons, neutrinos, and gravitational waves (figure 3c). Crucially, particle interactions at colliders behave the same as those inhabiting



(a) Early universe cosmology



(b) Dark matter



(c) Multi-messenger astronomy

Figure 3: **Particle physics connections with astrophysics and cosmology**. (a) History of the universe connecting particle physics with the hot Big Bang. (b) The Bullet Cluster provides evidence for dark matter via gravitational lensing (purple) displaced from the hot gas via X-rays (pink) [4]. (c) Artist's impression of an astrophysical multi-messenger event producing gravitational waves, photons, neutrinos, and protons. Images: CERN [5], NOIRLab, ESA/NASA/CXC/CfA/M.Markevitch/STScI, Magellan/U.Arizona/D.Clowe/ESO WFI.

distant galaxies and the early universe. This central fact arises because particles are excitations of the same quantum fields. This opens connections applying particle physics to exciting contemporary research areas in multi-messenger astrophysics.

The strong and weak nuclear forces also underpin stellar astrophysics. The solar proton–proton chain burns hydrogen into helium $2p+2e^- \to {}^4_2\mathrm{He}+2\nu_e$, overcoming proton electrostatic repulsion. A key process fuses two protons (hydrogen nuclei) into a deuterium nucleus of one proton and neutron:

$$p+p \to {}^2_1\mathrm{D}+e^+ +\nu_e. \tag{1.5}$$

This weak force mediates this crucial step and is the dominant factor limiting it rate in the

Sun. Its weakness compared to the strong force binding nucleons prevents the Sun burning out faster than giga-years. This enables planets to form and life to get started on Earth.

1.2 Applications and society

How does fundamental physics benefit other disciplines and wider society? Often overlooked in textbooks, particle physics finds widespread applications from neighbouring scientific fields to the humanities. Let us briefly survey some examples. Perhaps you will be inspired to find breakthrough applications using Higgs bosons for the betterment of humanity.

Big computing and the Web

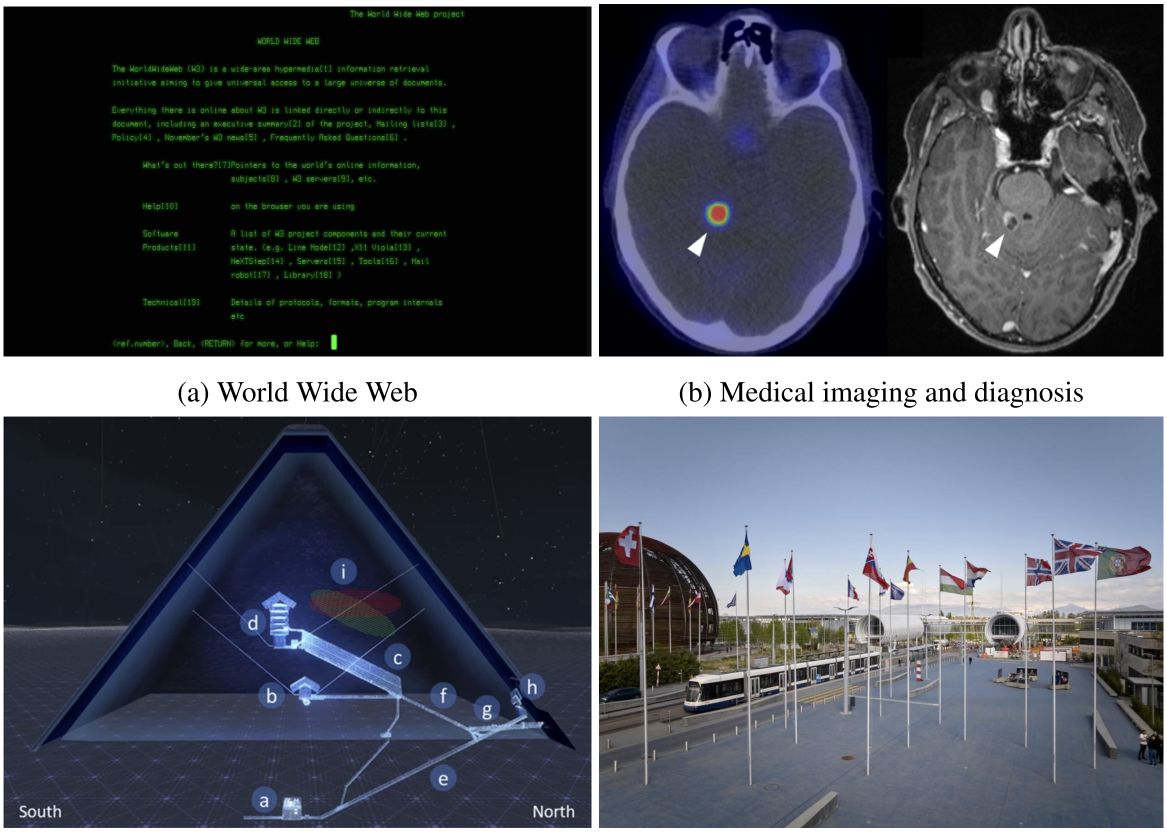
In 1989, Tim Berners-Lee presented an innocuous-sounding document entitled "Information Management: A Proposal" to CERN [7]. He sought to answer a frequently asked question of project management:

> "Yes, but how will we ever keep track of such a large project?"

The document was to convince CERN of the value of the World Wide Web. This serves to distribute information around the globe, which CERN adopted (figure 4a) and still uses today. This has had a truly transformative impact on society, to say the least, notably in the ubiquity of "`www`" prefixing website addresses. The development of an international Grid of high-performance computing was necessary for "big data", processing vast volumes of data and simulation for particle physics experiments.

Medical diagnosis and therapy

Scientists realised the significance of X-rays to medical diagnosis almost immediately after its discovery. By contrast, anti-matter and nuclear magnetic moments were esoteric curiosities in the early twentieth century with unclear utility. Today, they improve the human condition no less. Medical imaging relies on the flashes of gamma rays from anti-matter annihilating with matter in positron emission tomography (PET), while gyromagnetic protons enable magnetic resonance imaging. Precise imaging is vital in fields such as oncology [10] in cancer diagnosis and therapy (figure 4b). Cyclotrons originally developed to accelerate and study hadrons are now mainstays at hospital, who are the largest customers of particle accelerators. These accelerators create radioactive isotopes for medical imaging and novel radiotherapies such as cutting-edge proton therapy.



(a) World Wide Web

(b) Medical imaging and diagnosis

(c) Muon archaeology

(d) International relations

Figure 4: **Societal benefits and applications of particle physics**. World Wide Web for distributed information and computing, antimatter and nuclear magnetism applied to positron emission tomography (PET) and magnetic resonance imaging (MRI), cosmic-ray muon imaging revealing a previously unknown void in Khufu's Pyramid, and CERN as a nexus for international cooperation and cultural exchange. Images: CERN [8, 9], Cancer Imaging [10], Nature Communications [11].

Muography for archaeology and volcanology

An innovative interdisciplinary application of particle physics is muography: imaging using muons. As the flux of cosmic-ray muons is constant and well-known, it is possible to image the interiors of large stationary objects by measuring the scattering of incident muons. This recently revealed a previously unknown void in archaeological studies of Khufu's Pyramid, Egypt [11, 12] (figure 4c). This technique is also applied to probe the interior of active volcanoes. This includes the MUon RAdiography of VESuvius (MURAVES) experiment for Mount Vesuvius [13], a famously hazardous volcano given its proximity to Naples, Italy.

International relations

The European Laboratory for Nuclear Research (CERN) was established in 1954 as an international nexus for particle physics in the wake of the Second World War. It is now a model for peaceful international collaboration pursuing goals no single nation can accomplish alone (figure 4d). This is a remarkable feat of international relations and scientific diplomacy even during the Cold War.

In the arena of academic publishing, CERN brokered a landmark agreement called SCOAP^3 [14] with twelve journals, enabling nearly all particle physics publications to be open access since 2014. At a more individual level, these relations enable enriching research abroad experiences e.g. the CERN Summer Student Programme[18] to foster mutual understanding of diverse cultures and nationalities.

International cooperation towards a common goal has never been more important for tackling environmental problems that transcend borders from climate change to biodiversity degradation. This has become a heightened focus in the fundamental physics community given its enduring reliance on resource-intensive international facilities and datasets [15]. CERN has started publishing reports on its goals towards sustainable research practices [16]. No doubt these multi-disciplinary challenges facing our planet will endure in the future of international relations and particle physics.

Economic investment

The long-term benefits of research in fundamental sciences (typically needing decades) is often in tension with short-term timescales (typically a few years) that legislative agendas and research grants demand. By definition, it is challenging to predict what exploratory "blue skies" research will discover or how it will benefit society. Nonetheless, history illuminates an enviable track record.

As the story goes, William Gladstone was the Chancellor of the Exchequer[19] asked what the practical utility of electromagnetism was to Michael Faraday in the 1850s, who apocryphally replied[20]:

> "Why, sir, there is every probability that you will soon be able to tax it!"

From electric lighting and radios to computers and transportation, it would certainly be beyond their wildest dreams what technology is made possible. It would certainly be interesting

[18] https://home.cern/summer-student-programme

[19] The head of the British treasury.

[20] https://www.oxfordreference.com/display/10.1093/acref/9780191826719.001.0001/q-oro-ed4-00004273

to estimate of the tax revenue and business generated worldwide relying on electricity alone compared to the nineteenth century cost of electromagnetism research. With hindsight, this would likely qualify as a robust investment of public resources.

Cultural monuments

Understanding our origins in the cosmos is the bedrock of human culture, inspiring countless students to pursue the sciences. The Standard Model is a monument to human creativity akin to artistic and symphonic masterpieces admired across the centuries. Witnessing the scale of today's particle physics experiments inspires awe just as the engineering feats of ancient pyramids and medieval cathedrals.

One memorable exchange in the bid to fund a new accelerator was Robert Wilson's testimony at Congress in 1969, when US Senator John Pastore inquired:

> "Is there anything connected with the hopes of this accelerator that in any way involves the security of the country?"

Wilson replied rather eloquently[21]:

> "No, sir, I don't believe so... It has to do with: are we good painters, good sculptors, great poets? I mean all the things we really venerate in our country and are patriotic about. It has nothing to do directly with defending our country except to make it worth defending."

Congress subsequently approved funding for the construction of the Fermi National Accelerator Laboratory (Fermilab) just outside Chicago, Illinois.

Scientific literacy

Studying particle physics develops rigorous scientific literacy. These experiments embody exceptionally high levels of systematic control, validation, and reproducibility. Particle physicists set high standards for statistics both because the quality of data and large sample sizes enable such rigour. This training facilitates such individuals to interpret and evaluate the robustness of experiments and statistical analysis in literature from other disciplines. Quantitative reasoning, mathematical modelling, empirical inquiry, and critical thinking open successful careers in computational modelling, data science, finance, medical physics, school education, engineering, and instrumentation. Training such an advanced scientific workforce is a clear near-term benefit for modest investment of public funds.

[21] https://www.aps.org/archives/publications/apsnews/201804/history.cfm

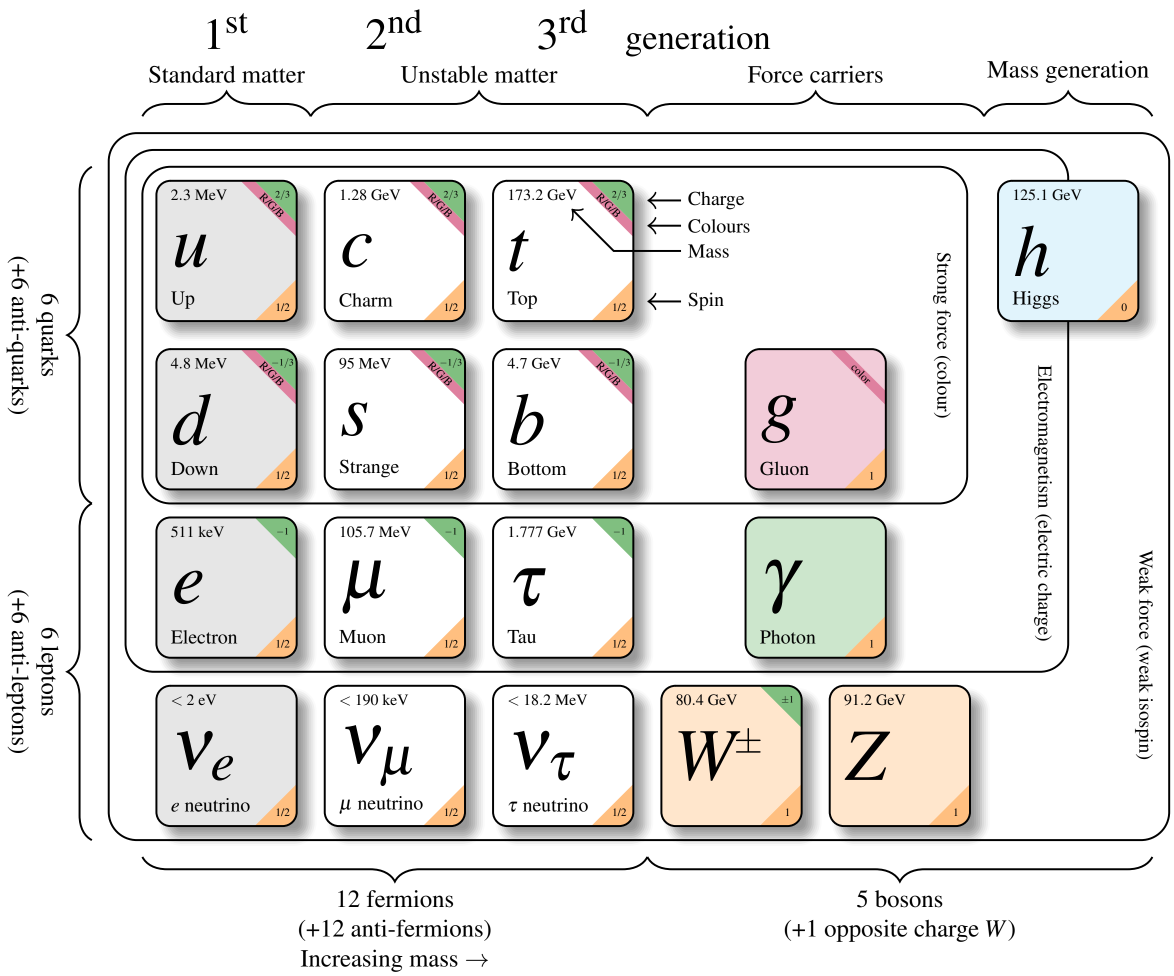


Figure 5: **Standard Model particle content and properties**. This diagram graphically summarises the experimentally known fundamental particles of matter and forces in the Standard Model. Image: adapted from Ref. [17].

1.3 Standard Model in brief

Sufficiently motivated, let us study the Standard Model. Just like arriving disorientated at an unfamiliar city, it is helpful to acquire a map whether for self-exploration or a guided tour. Figure 5 displays the particle content with their main quantum numbers such as mass and charge that underpin their interactions. This course provides a guided tour of the different parts of the SM, and this serves as the initial map to help orient you.

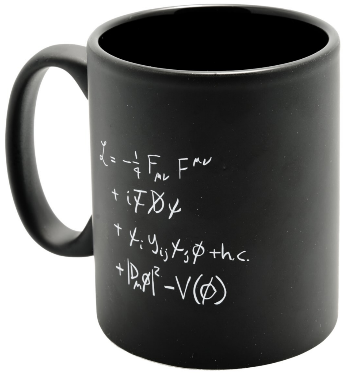
Particle content

The known matter in the universe comprises spin-1/2 particles called fermions categorised into quarks and leptons:

- There are 6 types of **quarks**. The different types are called their **flavour**. Quarks have fractional electric charge in units of 1/3 of the electron's charge.
- The up and down quarks are called the first **generation** of matter, which make up the neutrons and protons in everyday atoms. There are additional generation that have identical properties to the first generation of quarks, except they have heavier masses: these are the charm, strange, bottom and top. Quarks interact with the strong, weak and electromagnetic forces.
- There are 6 types of **leptons**, also called their flavour. The electron is familiar from atoms, and the (anti)electron neutrino emitted in beta decay.
- Leptons have a similar structure to quarks, where there are two copies of the first generation, which again have identical properties except their heavier mass. The heavier generations are the muon and tau-lepton, paired with their corresponding neutrinos. Leptons do not interact via the strong force.
- All the electrically charged quarks and leptons have **antiparticles** that carry opposite charge. Currently, not as much is known about the neutrinos: whether neutrinos have corresponding antiparticles, and only the mass differences between three neutrinos are measured, and there is no lower bound on the absolute neutrino mass.

Fundamental interactions are mediated by bosons with integer spin:

- **Electromagnetism**: this is mediated by a massless spin-1 boson called the **photon**, the quanta of light. These interact with all electrically charged particles.
- **Strong force**: this is mediated by a spin-1 boson called the **gluon**. These interact with all particles with colour charge, namely the quarks (and other gluons).
- **Weak force**: this is mediated by spin-1 bosons called the $W^{\pm}$ and Z bosons, or **massive gauge bosons**. These interact with particles that carry weak isospin charge, which all left-handed fermionic matter carries.
- **Mass generation**: at a fundamental level, mass is a unique manifestation of an interaction with a spin-0 boson called **Higgs boson**. This particle was observed in 2012 at



$$\begin{aligned}
\mathscr{L} = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} && \text{Force carriers} \\
&+ i\bar{\psi}\gamma^{\mu}D_{\mu}\psi && \text{Matter–force interactions} \\
&+ \psi_i y_{ij}\psi_j\phi && \text{Matter masses \& flavour mixing} \\
&+ |D_{\mu}\phi|^2 && \text{Force carrier masses} \\
&- V(\phi) && \text{Higgs mass \& self-interactions}
\end{aligned} \tag{1.6}$$

Figure 6: **Standard Model Lagrangian art**. This summarises the mathematical structure of matter and forces. The role of each term is heuristically described and unpacking its details is a central goal of this class. The mug is available as CERN merchandise and this Ref. [18] provides a lucid account accessible for high-school teachers and students. Image: CERN.

the Large Hadron Collider and it is the physical manifestation of the scalar field that generates mass for the quarks, charged leptons along with the $W^{\pm}$ and Z bosons.

- **Gravity**: this is mediated by a spin-2 tensor field whose fluctuations are gravitational waves recently detected in 2016. The LHC probes energy regimes far below the Planck scale $E \ll M_{\text{Planck}} = \sqrt{\hbar c/G} \approx 10^{19}$ GeV. The strength of gravitational interactions has at least $1/M_{\text{Planck}}$ suppression, rendering negligible impact on particle physics.

Underlying principles

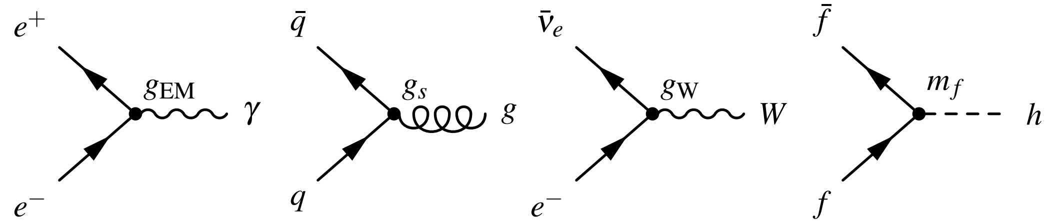
It is worth emphasising that particle physics is much more than discovering elementary particles. Indeed chemistry is no longer about discovering new chemical elements, but it remains an active and extraordinary discipline. Underpinning the periodic table of chemical elements are organising principles of quantum mechanics for its structure. Figure 6 illustrates this organising principle more mathematically. This shows the SM Lagrangian that fits neatly onto the widely-seen mug available at the CERN store. It appears remarkably simple because it is highlighting the mathematical *structure*, suppressing many details. This structure already shows how the wildly different observed behaviour of the fundamental forces actually share similar underlying physics. Figure 7 makes this similarity pictorially manifest. All four interaction diagrams share the same graph structure: two solid lines (fermionic matter and antimatter) meeting with one squiggly (spin 1) or dashed (spin 0) line at a vertex. As mathematical art, there is of course much to unpack in figure 6 but this coffee mug art serves as a helpful memory aid as this course progresses.

Major discoveries in fundamental physics actually did not involve new particles. They instead unveiled new principles overturning prevailing paradigms of nature:

- **Antimatter**. The Dirac equation predicted the existence of a completely new kind of matter that can annihilate with matter into massless photons.
- **Symmetry and conservation laws**. Noether's theorem relates fundamental symmetries to conserved quantities, namely spacetime symmetry and local (gauge) symmetry.
- **Broken discrete symmetries**: long-cherished symmetries of nature, namely parity and charge–parity conjugation, are mysteriously broken in weak interactions.
- **Forces as geometry**. All the fundamental forces are gauge theories, where a local symmetry dictated by a Lie algebra has a geometric interpretation. Gauge theory of forces have a remarkably similar structure to general relativity describing gravity.
- **Vacuum is dynamical**: the vacuum in classical physics is a static and empty, devoid of anything. Particle physics completely upends this idea, revealing that the vacuum is a teeming sea of particles and antiparticles popping in and out of existence.
- **Confinement and asymptotic freedom**: only one fundamental force in nature exhibits this enigmatic feature that binds nuclei together.
- **Mass as breaking of local symmetry**: particle physics radically changes our picture of inertial mass as enigmatically tied to interactions with a scalar field condensate via the Brout–Englert–Higgs mechanism.
- **Quantum field theory**: this is the theoretical framework of particle physics. Fields enable theories to eschew instantaneous action at a distance: they are manifestly local and causal. Upon quantisation, particles are excitations of the same fields permeating the whole universe. Particle physics tests many salient features of QFT.

Instruments enabling discoveries

New experimental instruments and techniques are pioneered in tandem with particle physics discoveries. We can now test the heaviest Standard Model particles and interactions with remarkable compatibility between theory and experiment (figure 9). These notes endeavour to integrate the key experimental methods for revealing the Standard Model:



(a) Electromagnetism (b) Strong force (c) Weak force (d) Higgs interaction

Figure 7: **Standard Model interactions vertices**. These graphs show fermionic matter (solid straight lines) interacting with bosonic force carriers for the fundamental forces of particle physics.

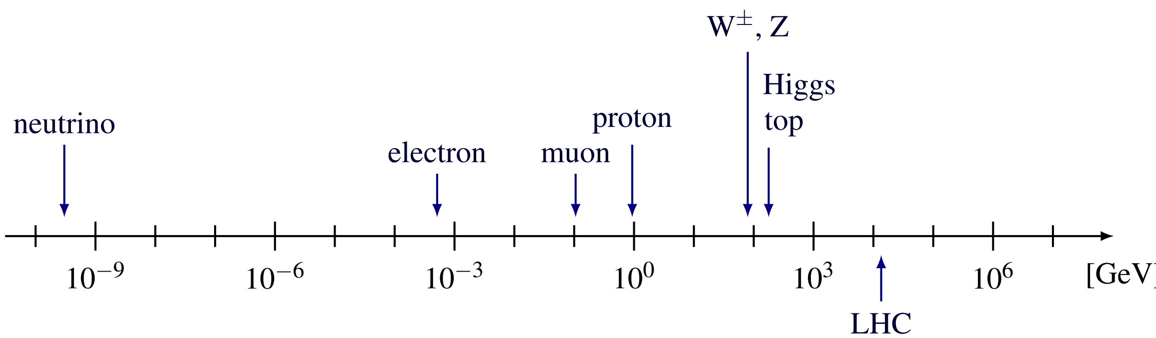


Figure 8: **Masses of selected Standard Model particles**. Also marked is the Large Hadron Collider (LHC) centre-of-mass energy. Adapted from tikz.net.

- **Radiation from the ground and sky**: the discovery of radioactivity and cosmic rays gave physicists a natural source of energetic particles. These played a key role in the foundations of nuclear and particle physics, leading to the discoveries of the strong and weak forces alongside the positron, muon, pion and kaons.

- **Particle accelerators**: Accelerators now reach far higher energies and intensities than natural sources of radiation. This enables the creation of new particles in the controlled environment of in laboratories, ushering many major discoveries. We will briefly touch upon the first linear and circular accelerators, focusing on examples to illustrate their importance for particle physics discoveries. Historically integral to particle physics, accelerator physics has developed into its own vibrant subfield in recent decades.

- **Particle detectors**: inventing new methods to detect particles flying out of colliders is pivotal to discoveries. The principles underpinning the revolutionary cloud chamber and Geiger counter remain largely how we detect particles today. We will also briefly

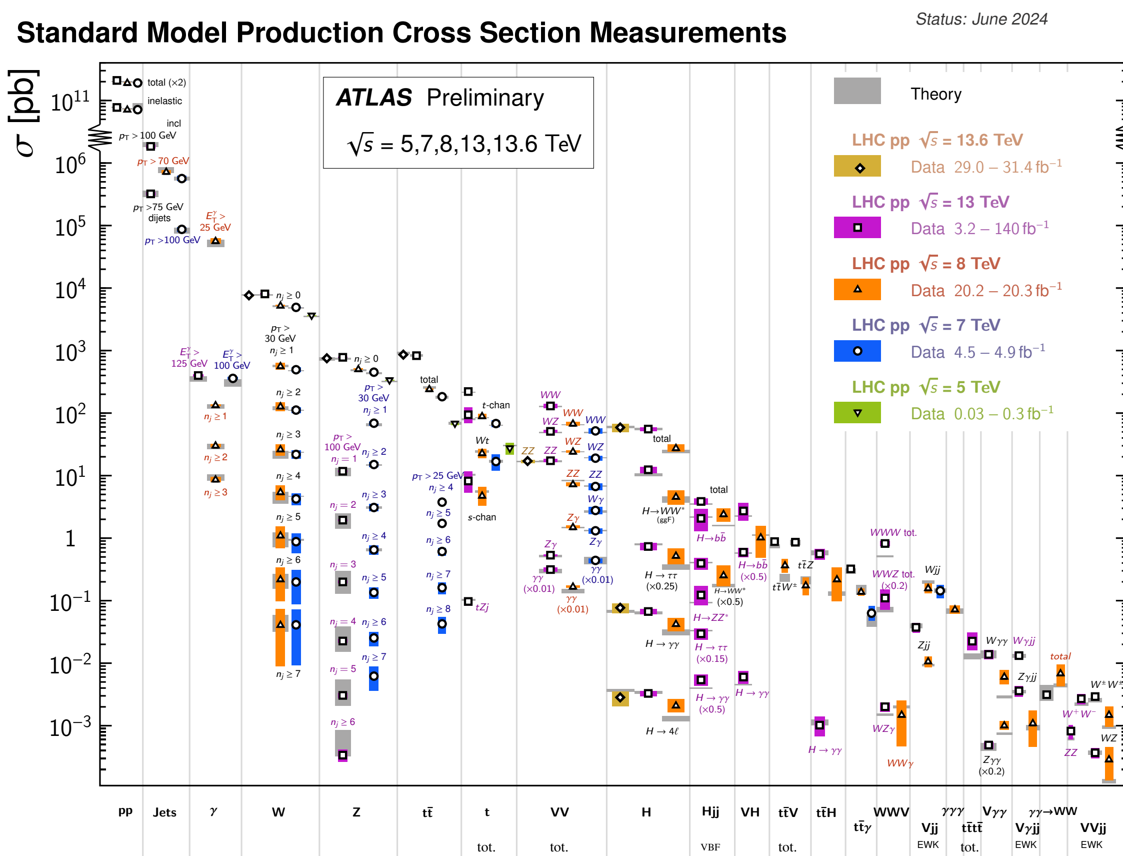


Figure 9: **Standard Model cross-sections at LHC**. Measurements and predictions by the ATLAS Collaboration compared to theoretical calculations for a wide variety of Standard Model processes across different LHC centre-of-mass energies $\sqrt{s}$ [19].

look at photographic emulsions and bubble chamber. After this, we study the general-purpose ATLAS and CMS detectors at the LHC in the context of the electroweak scale.

- **Electronics and computing**: initial particle physics experiments were meticulously by humans recording scintillations by hand or scanning photographs in optical microscopes. As particle event rates and semiconductor technology advanced, particle physics thrived off electronic automation and large-scale computing. We will only mention these topics in passing.

2 Foundational experiments

We start by illustrating landmark experimental discoveries from the late 19th and early 20th centuries. This provided the early evidence for subatomic states that form the foundations of particle physics:

$$\text{electron } (e^-), \text{ neutrino } (\nu_e), \text{ proton (p)},$$
$$\text{neutron (n), positron } (e^+), \text{ muon } (\mu^\pm).$$

These brief accounts in this section far from replace the full history of science, but serve as important reminders for the scientific method of discovery. Experiment repeatedly surprised us. Scientific discoveries are often initially confusing and serendipitous, relying on a multitude of observations and persistent investigation usually over decades to elucidate.

2.1 Radiation and radioactivity

Cathode rays and the electron

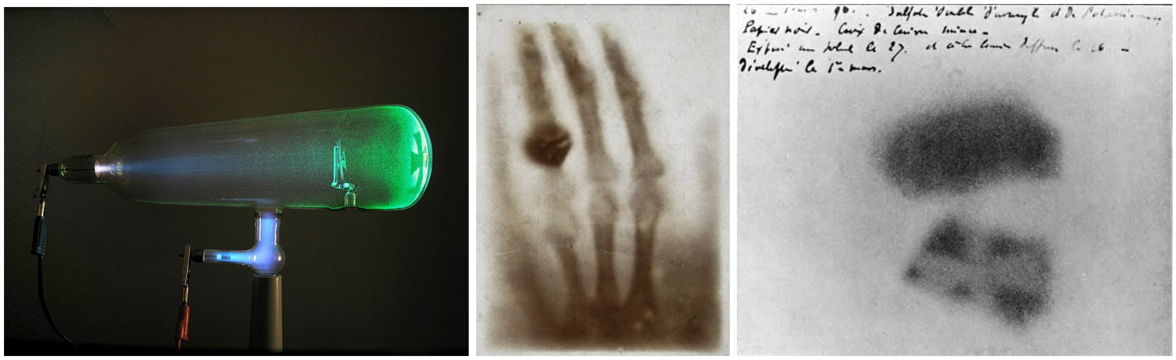
The first particle accelerator and signature of electrons were observed in evacuated **Crookes tubes** conceived in 1869 (figure 10a), which are cathode-ray tubes. Electromagnetism was still nascent (Maxwell's equations were written in 1861), but William Crookes, Johann Hittorf and Eugen Plücker found that current can flow from a cathode to anode in the evacuated glass tube, suggesting charged rays. Crucially, they caused glass to fluoresce and phosphorescent materials illustrated their straight-line paths, with metal crosses blocking these rays to cast shadows. Hendrik Lorentz wrote down explicitly the force on moving charges, which can combine with Newton's second law to give

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\ddot{\mathbf{x}} = \mathbf{f} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}). \tag{2.1}$$

This shows that particles with the same charge-to-mass q/m ratio behave the same under electromagnetic fields. One can apply the centripetal force $f = mv^2/r$ from the magnetic field onto cathode rays balanced by the electric force $f = qE$ to relate q/m to the measured E and B field strengths:

$$\frac{q}{m} = \frac{1}{r^2}\frac{E}{B}. \tag{2.2}$$

J. J. Thomson performed this in the classic experiment at Cambridge in 1897, measuring the charge-to-mass ratio q/m for cathode rays to be a constant $q/m \approx -1.7 \times 10^{11}\ \mathrm{C\,kg^{-1}}$, three orders of magnitude larger than that of a hydrogen ion.



(a) Crookes tube: cathode rays (b) Röntgen: X-rays (c) Becquerel plate: radioactivity

Figure 10: **Discovery of radiation and radioactivity**. Early nineteenth century ray experiments observing: cathode rays emitted in an evacuated Crookes tube (electrons), photographic plates exposed to X-rays (photons), and radioactivity from uranium salts on photographic plate (helium nuclei and electrons). Images: D-Kuru/Wikimedia Commons, Nobel Prize/public domain, Muséum national d'histoire naturelle (Paris).

Of course, measuring this q/m ratio implies either the charge could be 1000 times larger, or the mass were 1000 smaller than hydrogen ions. The latter seemed less implausible, but the only way to experimentally disambiguate this was to directly measure the cathode ray charge. Thomson saw these charged particles cause water vapour to condense into droplets, a primitive cloud chamber. Balancing their gravitational force against an electric field determines the charge to be around 10^{-19} C, the same as hydrogen ions. This was the landmark discovery of the electron.

Like many of his contemporary scientists, Wilhelm Röntgen was also playing around with cathode-ray tubes. He covered one with black card and saw it caused nearby phosphorescent screens to glow green even over a metre away. He had discovered the emission of new invisible rays that could mysteriously travel farther than cathode rays in air, and called them X-rays. He found that they exposed photographic plates and famously imaged (figure 10b) the bejewelled hand of his wife, Anna Bertha Ludwig, which he presented to Ludwig Zehnder at Freiburg in January 1896. This serendipitous discovery caused tremendous excitement and opened the field of X-ray medical imaging that is now ubiquitous in modern society.

Radioactivity

Henri Becquerel was no less fascinated by these newly discovered X-rays and how materials phosphoresce. He was investigating phosphorescence from his uranium salts, which he thought arose from absorbing solar radiation before re-emitting this energy as X-rays that expose photographic plates.

As the famous story of accidental discovery goes[22], the weather soon turned overcast in Paris and Becquerel stored the uranium salt and photographic plate in a dark drawer away for experiments another sunny day. Whether a moment of inspiration or impatience, he developed the photographic plate anyway on 1 March 1896. Remarkably, he instead saw a high-contrast cross imaged on the plate and reported it to the Academy of Sciences the next day (figure 10c), showing uranium emitted radiation despite not absorbing light from the Sun. Through further investigation, he later found that unlike neutral X-rays, electromagnetic fields could bend rays emitted from uranium salts.

In parallel, Marie Skłodowska-Curie and Pierre Curie discovered more radioactive elements: thorium followed by polonium and radium. In 1900, Paul Villard discovered a new radiation from radium identified as gamma rays. Ernest Rutherford classified these wide-ranging reports of radiation into three categories by their empirical properties of material penetration and ionisation power: alpha α, beta β, gamma γ, X-rays.

These luminaries formulated the fundamental ideas of radioactivity. The decay rate was related to the quantity of radioactive material. In a time interval $\mathrm{d}t$, the decrease in the number of particles $\mathrm{d}N$ is given by the number of such particles N multiplied by its quantum mechanical transition rate $\Gamma/\hbar$ ($\hbar$ being the reduced Planck's constant)

$$\mathrm{d}N = -N(\Gamma/\hbar)\mathrm{d}t. \tag{2.3}$$

Integrating gives the exponential decay formula that is ubiquitous in particle physics

$$N = N_0 \mathrm{e}^{-\Gamma t/\hbar}, \tag{2.4}$$

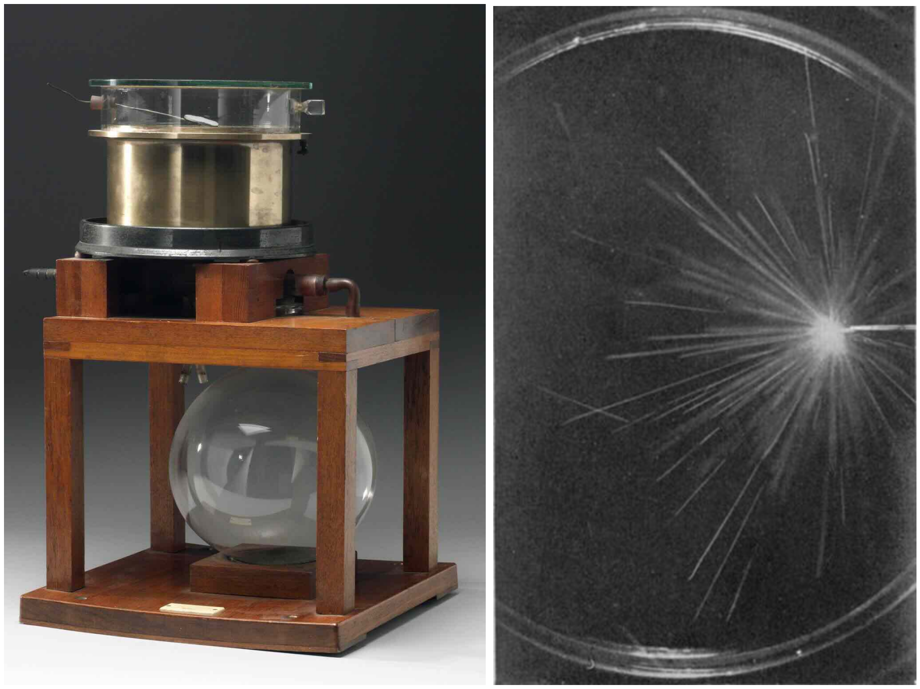
where we can define the proper lifetime $\tau = \hbar/\Gamma$. In cases where a particle can decay via multiple processes $\{i\}$, where we can ascribe each process i to a partial decay rate Γ_i, and the total decay rate is given by the sum $\Gamma = \sum_i \Gamma_i$.

These monumental discoveries established the instability of matter that can transmute into different elements. While far from obvious then, it also provided the first indirect hints of new fundamental interactions: the strong and weak nuclear forces.

Ionisation and cloud chambers

Meanwhile, groundbreaking instruments were invented and developed to detect radiation. In 1908, Hans Geiger conceived the concept of the **ionisation chamber** [21], using gases to de-

[22] https://www.aps.org/apsnews/2008/02/becquerel-discovers-radioactivity



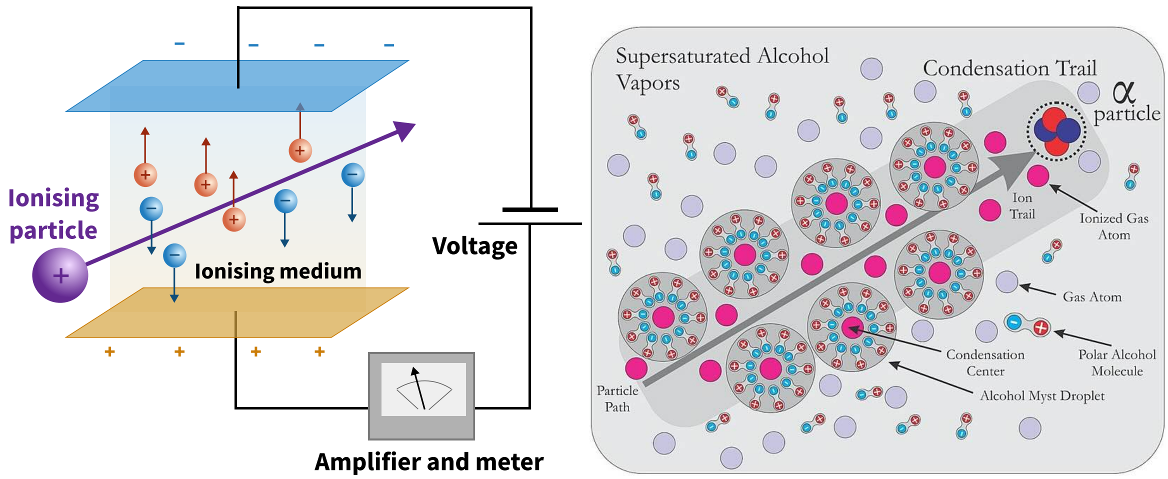
(a) Wilson cloud chamber (b) α-particles [20]

Figure 11: **Wilson expansion cloud chamber apparatus**. The cloud chamber is cylinder on the upper left around 15 cm in diameter. Also displayed is a 1912 photograph of alpha particles from a radium source. Left: Science Museum/UCL.

tect ionising particles. The principle of operation is shown in figure 12a, where a charged particle ionises a medium, inducing a measurable electrical voltage. The underlying physics of ionisation is an incident charged particle imparting sufficient energy to eject an electron originally bound to an atom. The various binding energies of atoms motivate different choices of ionisation media. This is still the fundamental principle used in charged-particle detectors today, including the famous Geiger counter.

Meanwhile, another pivotal instrument developed concurrently was the **expansion cloud chamber**. Inspired by metrological phenomena on a visit to Ben Nevis in Scotland, Charles T. R. Wilson wanted to study cloud formation in the laboratory upon returning to Cambridge. Little did he know this would make profound discoveries not in atmospheric but subatomic physics. Wilson realised by expanding a chamber of humid air, super-saturated vapour condenses in the absence of dust along the path of ionising particles.

By 1911, Wilson perfected the expansion cloud chamber [20, 22], where a piston is pulled to suddenly expand the condensation volume. This expansion is synchronised to a



(a) Ionisation chamber (b) Diffusion cloud chamber

Figure 12: **Ionisation principles of particle detection**. A charged particle enters the ionising medium, causing this the atoms to ionise along its path. This causes a current to flow between a cathode and anode biased by a voltage, which can be amplified and measured. This principle remains the basis for how today's charged-particle detectors function. Right image: Kotarak71/Wikimedia.

flash of light and camera to record the resulting particle tracks. This enabled taking stunning photographs of charged particles (figure 11a) that would have an illustrious record of groundbreaking discoveries in ensuing years. The bubble chamber, a mainstay for particle detection in the 1960-70s, is based on similar principles but with liquid turning into gas upon ionisation (section 8). A popular education and outreach tool today is the diffusion cloud chamber[23], where a future iteration of this class with sufficient resources may consider building one as a class project.

Indirect evidence for neutrinos

At this point, it is worth mentioning the indirect evidence for neutrinos in beta decay. An everyday example of β^- decay is potassium-40 present in e.g. bananas, which can decay to calcium-40 via emission of an electron e^- and electron anti-neutrino $\bar{\nu}_e$:

$$^{40}_{19}\mathrm{K} \rightarrow ^{40}_{20}\mathrm{Ca} + e^- + \bar{\nu}_e. \tag{2.5}$$

[23] https://scoollab.web.cern.ch/cloud-chamber: CERN S'Cool LAB has a DIY manual for how to build one using isopropyl alcohol and dry ice.

Nuclear beta decay involves a change of proton number Z leaving the nucleon/mass number A unchanged. This is equivalent to a neutron decaying into a proton:

$$n \to p + e^- + \bar{\nu}_e. \tag{2.6}$$

Historically, there are two classic arguments for the existence of a third unobserved particle in β^- decay, now known as an anti-neutrino:

1. **Energy spectrum of electron**. Working in the rest frame parent nucleus, the resulting products of a two-body decay $\mathrm{X} \to e^-\mathrm{Y}$ should be equal and opposite in momentum. The emitted electron E_e should have a well-defined peak related to the masses of the parent and child particles given by energy-momentum conservation of equation (3.15):
$$E_e = \frac{m_X^2 - m_Y^2 + m_e^2}{2m_X}. \tag{2.7}$$
However, his process is not observed:
$$^{40}_{19}\mathrm{K} \to {}^{40}_{20}\mathrm{Ca} + e^- \quad \text{2-body decay not observed.} \tag{2.8}$$
This is inferred from the observation by James Chadwick that beta decay has a spectrum of energies. This is the hallmark of a three (or more) body decay.

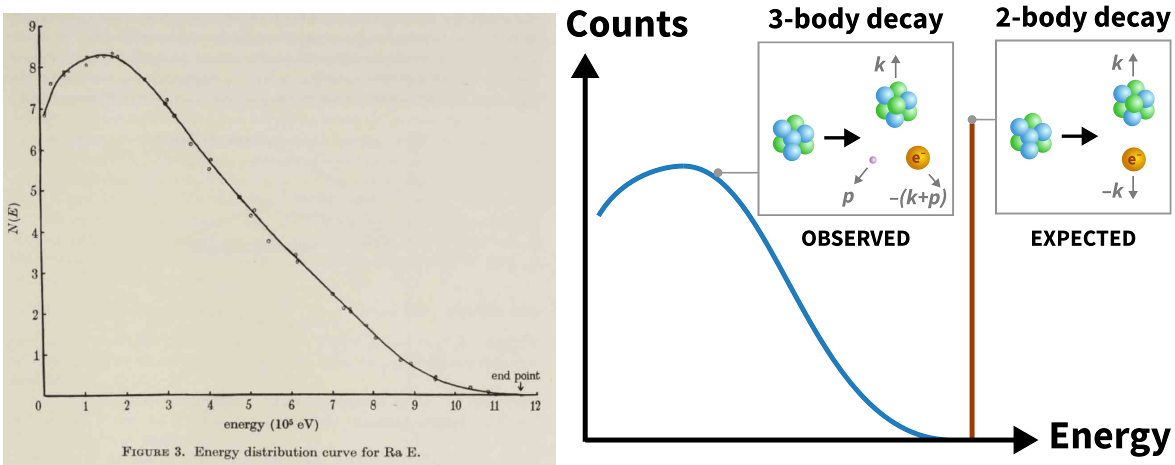
2. **Conservation of angular momentum**. The initial neutron on its own is spin-$\frac{1}{2}$. If a proton and electron, both spin-$\frac{1}{2}$ states, were the only final states, the total final angular momentum numbers[24] are $S = 0$ or 1. Angular momentum conservation require an additional spin-$\frac{1}{2}$ particle to exist. Three spin-$\frac{1}{2}$ particles couple together to give total angular momentum numbers of $S = \frac{1}{2}$ or $\frac{3}{2}$.

The continuous spectrum of electron was historically a vexing mystery about β^- decay. If you measured the electron energy by bending them through a magnetic field and noting their radii, the spectrum is continuous has a maximum energy at 1.3 MeV like those from figure 13a. To resolve this apparent non-conservation of energy and momentum, Wolfgang Pauli proposed in 1930 that an invisible particle, which we now know to be the electron antineutrino $\bar{\nu}_e$, is carrying away the electron energy:

$$n \to p + e^- + \bar{\nu}_e \quad \text{3-body decay is observed.} \tag{2.9}$$

This explains how the available kinetic energy is shared between the electron and anti-neutrino in three-body decays. This constitutes indirect evidence for the existence of neutrinos. It took the advent of nuclear reactors and the Cowan–Reines experiment to enable direct detection of anti-neutrinos in 1956 (section 12.2).

[24]Recall the rules of angular momentum addition regarding the quantum numbers. For two particles with spins s_1 and s_2, the total spin numbers are $S = |s_1 - s_2|, |s_1 - s_2| - 1, \ldots, s_1 + s_2 - 1, s_1 + s_2$.



(a) Beta decay measurements [23]. (b) Energy: 2 vs 3 body decay.

Figure 13: **Beta decay spectrum**. The early expectation of a two-body decay sees the child nucleus and electron having equal and opposite momentum, resulting in a sharp peak in energy from equation (2.7). The observed three-body decay features an additional anti-neutrino with momentum **p** smearing out the observed spectrum into a continuum.

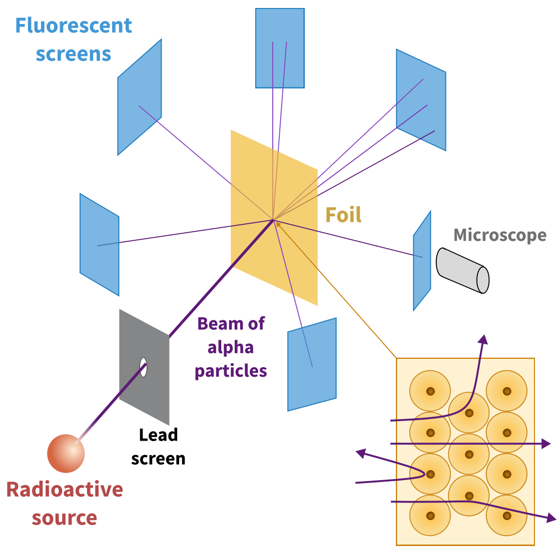
2.2 Nuclear scattering

When we first hear the famous story of Hans Geiger, Ernest Marsden, and Ernest Rutherford discovering the atomic nucleus and proton at Manchester, we naturally ask what made them fire alpha particles at metal foils in the first place? Perhaps it was just a benign task to keep the new student Marsden busy? There was little reason to doubt J. J. Thomson's then prevailing plum-pudding model of electrons spread evenly throughout an atom.

It turns out they were actually investigating a seemingly unrelated detector imperfection. The new ionisation chamber Geiger invented in 1908 occasionally made unexplained erratic measurements from traversing alpha particles. This motivated more systematic characterisation of how alpha particles interact with matter, in another case of one advance opening further serendipitous discoveries.

Discovery of atomic nucleus

They designed their classic experiment, firing alpha particles from a radium source at metallic foil and measured how the alpha particles scattered across angles ϕ (figure 14a). They preferred gold foil due to its malleability and they could press it very thin. As a detector, they used a zinc sulfide screen that fluoresced upon being struck by scattered alpha particles. One



(a) Schematic of scattering experiment

610 Dr. H. Geiger *and* Mr. E. Marsden *on the Laws of*

TABLE II.

Variation of Scattering with Angle. (Collected results.)

I. Angle of deflexion, ϕ.	II. $\frac{1}{\sin^4 \phi/2}$.	III. Silver. Number of scintillations, N.	IV. Silver. $\frac{N}{\sin^4 \phi/2}$.	V. Gold. Number of scintillations, N.	VI. Gold. $\frac{N}{\sin^4 \phi/2}$.
150°	1·15	22·2	19·3	33·1	28·8
135	1·38	27·4	19·8	43·0	31·2
120	1·79	33·0	18·4	51·9	29·0
105	2·53	47·3	18·7	69·5	27·5
75	7·25	136	18·8	211	29·1
60	16·0	320	20·0	477	29·8
45	46·6	989	21·2	1435	30·8
37·5	93·7	1760	18·8	3300	35·3
30	223	5260	23·6	7800	35·0
22·5	690	20300	29·4	27300	39·6
15	3445	105400	30·6	132000	38·4
30	223	5·3	0·024	3·1	0·014
22·5	690	16·6	0·024	8·4	0·012
15	3445	93·0	0·027	48·2	0·014
10	17330	508	0·029	200	0·0115
7·5	54650	1710	0·031	607	0·011
5	276300	...	...	3320	0·012

(b) Deflection results [24]

Figure 14: **Manchester Geiger–Marsden–Rutherford experiment**. Schematic of experimental setup and deflection results supporting the subatomic nucleus theory and discovery of proton.

day, Rutherford suggested they see how many reflected back on itself and found a few out of 10,000 reflected back, which was a complete surprise.

Rutherford proposed a central positively charged nucleus with negatively charge surrounding it [25]. A textbook calculation of mechanics textbooks assuming Coulomb's law with conservation of energy and momentum gives the Rutherford differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{zZe^2}{4\pi\varepsilon_0} \frac{1}{4E} \right)^2 \frac{1}{\sin^4(\theta/2)}, \tag{2.10}$$

where ze (Ze) is the incident (target nucleus) electric charge, $E = mv^2/2$ is the incident alpha-particle kinetic energy, θ is the scattering angle[25] and $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the differential solid angle. Geiger and Marsden painstakingly counted scintillation hits in a darkened laboratory[26] on florescent zinc sulfide screens viewed through a microscope as they rotate through different angles to establish the high-statistics datasets. Figure 14b shows the results from the 1913 paper, where the meticulously collected data of scintillation hits $N(\theta)$ support the predicted $\sin^{-4}(\theta/2)$ dependence from Rutherford's calculations.

[25] In the historic papers (figure 14b), they use ϕ.

[26] Manually without digital electronics or computers. Scientists certainly have great patience.

Discovery of neutron

The story of the neutron discovery [26] starts with Herbert Becker and Walther Bothe in Berlin, 1930. They bombarded a beryllium target with alpha particles from a polonium source. They noticed an unknown form of highly penetrating radiation that was energetic and electrically neutral. Back then, the only known candidate for this mysterious neutral radiation was a gamma ray:

$$^{4}_{2}\alpha + ^{9}_{4}\mathrm{Be} \to ^{13}_{6}\mathrm{C} + \gamma \quad ? \tag{2.11}$$

In the ensuing years at Paris, Irène and Frédérick Jolio-Curie[27] (figure 15a) extended the Becker–Bothe experiment. They directed the invisible radiation from the bombarded beryllium target onto paraffin wax, which is a hydrocarbon C_nH_{2n+2} used in candles rich in hydrogen, and reported energetic protons being ejected [27]. Whatever this invisible radiation was, it was powerful stuff to knock out hefty protons. They had found evidence for the existence of a new kind of radiation, but still misidentified it as gamma rays. This was the experimental setup of what was happening:

$$\boxed{\begin{array}{c}\text{Polonium}\\\text{source}\end{array}} \xrightarrow{\text{alphas}} \boxed{\begin{array}{c}\text{Beryllium}\\\text{target}\end{array}} \xrightarrow{\text{neutrons}} \boxed{\begin{array}{c}\text{Paraffin}\\\text{wax}\end{array}} \xrightarrow{\text{protons}} \boxed{\begin{array}{c}\text{Ionisation}\\\text{chamber}\end{array}}$$

These pioneers had not realised they discovered the neutron, but there were intriguing reasons the scientific community were both reluctant and expecting it. Indeed upon hearing the Jolio-Curie experiment, Ettore Majorana purportedly exclaimed [26]:

> "These fools have discovered the neutral proton and are not aware of it."

Back in the 1920s, scientists already noticed atomic mass numbers A increased quicker than the proton number Z, which you can see on the first rows of the periodic table (figure 1). High school students today easily ascribe this mismatched progression to neutrons. So why was history slow to propose this? It was because there was a seemingly reasonable idea that atomic nuclei contained neutral bound states of one proton with one electron. This appeared sensible to explain how beta decay induced atoms to emit electrons with MeV energies, far higher than eV energies of electrons surrounding the nuclei.

James Chadwick had sought such a neutral bound nuclear state for many years while working at Cambridge. When he heard the results from the Joliot-Curie experiment in January 1932, Chadwick quickly reproduced their experiment with the interpretation of a new

[27] Irène Curie is the daughter of Marie and Pierre Curie, who married Frédérick Jolio, and pursued research efforts together from c. 1928.

(a) Irène & Frédérick Jolio-Curie (b) Ejected proton (c) Chadwick ionisation chamber

Figure 15: **Discovery of neutron via proton ejection**. The photograph displays the aftermath of alpha particles bombarding a beryllium target, producing an e^+e^- pair from gamma ray conversion together with the thick track showing a proton that had been ejected by a neutron. Chadwick used an ionisation chamber to detect the ejected protons. Images: Gallica and London Science Museum, Cambridge Cavendish Laboratory.

particle called the neutron [28]:

$$^4_2\alpha + {}^9_4\text{Be} \rightarrow {}^{12}_6\text{C} + {}^1_0\text{n}. \tag{2.12}$$

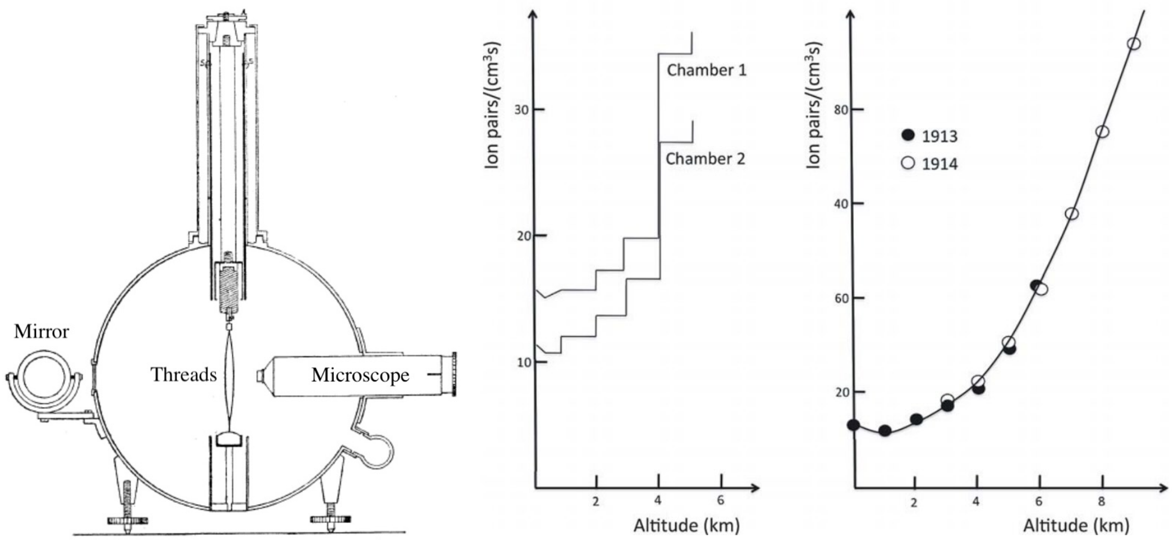
The ionisation chamber used by Chadwick is displayed in the Cavendish Laboratory museum at Cambridge (figure 15). While history of science often cites Chadwick discovering the neutron, the first empirical evidence was already present in the Becker–Bothe and Jolio-Curie experiments.

2.3 Cosmic rays

This class mostly highlights cosmic rays as a natural source of high-energy particles before the advent of accelerators and its central role in unveiling the Standard Model. Today, cosmic-ray physics is its own vibrant field and the High Energy Astrophysics (PHYS-GA 2050) classes cover its phenomena in greater detail.

The first indirect signature of cosmic rays, with hindsight, dates back to Charles-Augustin de Coulomb (1785) reporting spontaneous discharge of electrostatic devices no matter how well-insulated he made them[28]. These enigmatic observations were confirmed by Michael Faraday (1835), while Cano Matteucci (1850) and William Crookes (1879) observed how their rate decreased at lower atmospheric pressures [29].

[28] https://timeline.web.cern.ch/first-observations-spontaneous-discharge-electrometer



(a) Wulf electroscope (b) Hess and Kolhörster 1912–14 balloon data

Figure 16: **Historical measurements of cosmic-ray flux**. Electroscopes measure ionisation by observing metallic threads under tension during electrostatic discharge. This enabled balloon-bourne measurements by Hess (1912) and Kolhöster (1913), illustrating the increase in flux with altitude. Images: Wulf, Phys. Zeit. 1 (1909) 152, Ref. [29].

Once radioactivity was discovered, the expectation was that the radiation should decrease with altitude assuming rocks on Earth were the sole source. Individuals including Theodore Wulf[29] and Domenico Pacini [30] presented initial evidence for the altitude dependence of radiation above and below sea level, respectively. More portable and sensitive electroscopes to measure electrostatic charge were crucial advances, which Wulf improved by replacing conventional gold leaves with metalised threads (figure 16a).

In 1912, Viktor Hess initiated the pivotal balloon-bourne measurements using Wulf electroscopes up to 5 300 m [31] (figure 16b). Electroscopes were a common ways to measure the presence of charged particles during the development of electromagnetism, often comprising gold leaves or torsion balance that detects electrostatic repulsion. With further measurements at higher altitudes over 8 km by Werner Kolhörster in 1913–1914 provided further evidence, leading to the decisive discovery of cosmic rays.

Growing interest in mountaineering around this period certainly helped those who did not fancy venturing into hot-air balloons. Many groundbreaking cosmic-ray experiments were exposed to radiation at high mountains, where photographic emulsion plates can be left

[29] https://www.aps.org/apsnews/2019/08/wulf-publishes-evidence-cosmic-radiation

undisturbed for long exposures. We shall defer this discussion of pions and kaons discovered in cosmic rays to section 8 on the strong force.

Positron and antimatter discovery

Figure 17a shows the iconic positron e^+ track as the dark track inside a Wilson cloud chamber. Anderson constructed a $(17\text{ cm})^2 \times 3$ cm chamber and took several photos of cosmic rays in Pasadena, California. The dark horizontal rectangle in the middle is a lead plate that slows down the particle, where the tighter bend above than below the lead plate implies its trajectory must be upwards. One of the innovations of was to immerse the cloud chamber in a magnetic field for charge and momentum determination via the Lorentz force law (2.1). The relationship between the momentum magnitude p traversing perpendicular to a magnetic field B with bending radius R is given by

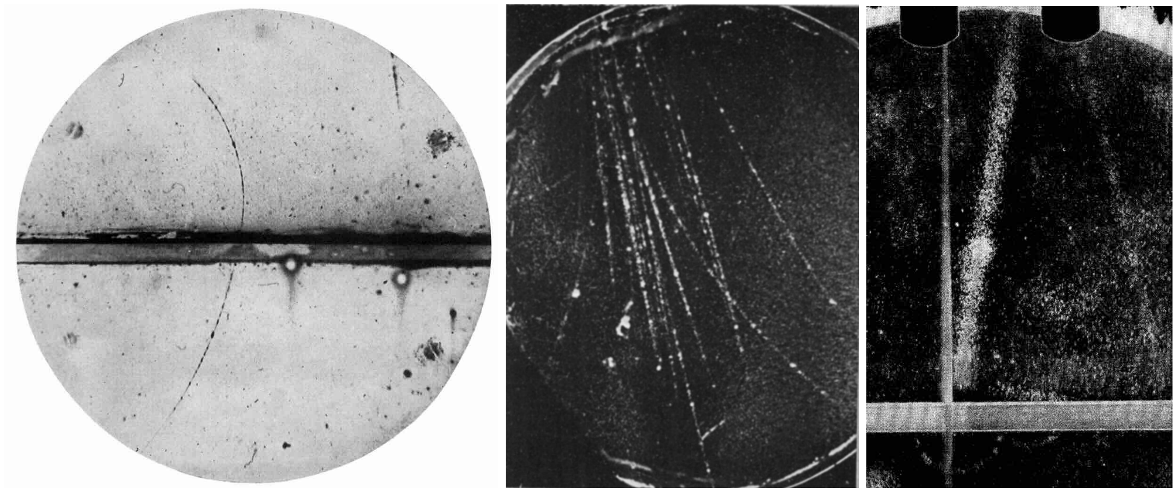
$$\frac{p}{\text{GeV}/c} = k\left(\frac{B}{1\text{ Tesla}}\right)\left(\frac{R}{1\text{ metre}}\right), \quad k \approx 0.3 \tag{2.13}$$

This serves as a remarkably ubiquitous formula for the bending of a relativistic charged particle in accelerators and detectors. Bending anti-clockwise in the 1.5 T magnetic field pointing into the page, the particle must be positively charged. By measuring the curvature, the deduced momentum is 63 MeV below and 23 MeV above the plate.

The only known positively charged particle at the time was the proton. By assuming a proton mass, one could deduce its energy from the measured momentum. Counting the droplets per unit length, this determined the particle ionisation properties and the mean path length based on its energy. Anderson noted that the path length was "at least ten times greater than the possible length of a proton path of this curvature". By accumulating several such tracks, the mass of the particle was determined to within 20% of the electron mass. This was the stunning discovery of the **positron** and **antimatter**.

Historically and even today, experimentalists and theorists did not always appreciatae each other's work even while neighbours. When Paul Dirac published his eponymous equation in 1928 (section 3.2), few took its negative energy states seriously as physical reality. Patrick Blackett and Giuseppe Occhialini were contemporaries of Dirac at Cambridge, but were busy inventing a groundbreaking new toy in 1931: the **counter-controlled cloud chamber**. They ingeniously combined a pair of Geiger-Muller tubes sandwiching above and below a Wilson cloud chamber.

Upon traversal by an energetic cosmic ray, the Geiger–Muller tubes send an electronic signal to **trigger** the cloud chamber piston and camera shutter: neat automation underpinning all triggers of today's experiments. By contrast, Anderson's cloud chamber randomly took



(a) Positron track [32] (b) e^-e^+ pairs [33] (c) Muon track [34]

Figure 17: **Cloud chamber discovers antimatter and muon**. (a) The dark line curving upwards anti-clockwise is the positron (B field into page). (b) The original caption writes two electrons bending clockwise, which are rather faint on the left, with two positrons bending anti-clockwise on the right of the image (B field into page). (c) The muon is the thick line.

pictures even if there were no energetic ionising particles, so he had to sift through thousands to find the positron. When Blackett and Occhialini heard Anderson's stunning discovery, they checked other photos from their new device and realised it had recorded positrons all along in great abundance (figure 17b). In a 1933 paper with a terribly understated title "Some photographs of the tracks of penetrating radiation" [33], they discovered e^-e^+ **pair production** via unmistakable mirror-image curving tracks.

In the backdrop of the positron discovery and their earlier experiments creating neutrons, the Joliot–Curies continued their pioneering work bombarding aluminium with alpha particles to create phosphorus:

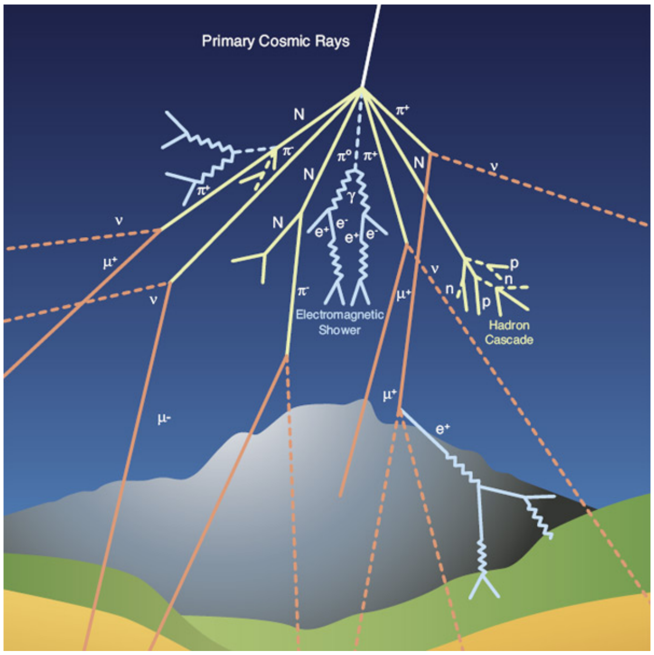
$$ {}^4_2\alpha + {}^{27}_{13}\mathrm{Al} \to {}^{30}_{15}\mathrm{P} + {}^1_0\mathrm{n}. \tag{2.14} $$

This isotope of phosphorus-30 is special because it decays via positron emission with a half-life of 2.5 minutes, providing first evidence for β^+ decay:

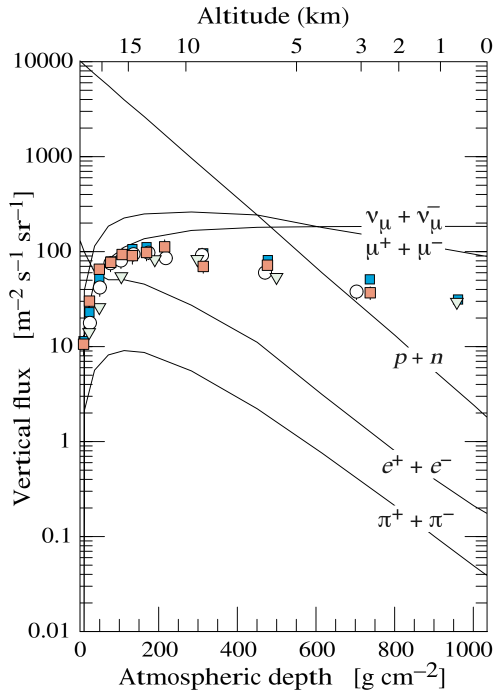
$$ {}^{30}_{15}\mathrm{P} \to {}^{30}_{14}\mathrm{Si} + e^+ + \nu_e. \tag{2.15} $$

This novel synthesis of radioactive elements emitting positrons marked the first **laboratory production of antimatter** in 1934[30]. It is also central to nuclear medicine for the creation of radioactive isotopes at hospitals.

[30]The Jolio–Curie pair received the 1935 Nobel prize in chemistry for this discovery, the same year the prize



(a) Cosmic-ray air shower CERN



(b) Flux composition with altitude [35]

Figure 18: **Cosmic-ray air showers**. Schematic drawing of extensive air shower induced by primary cosmic-ray striking the an atmopsheric nucleus. The contemporary flux vs altitude uses points to show various measurements of $E > 1$ GeV muons μ^-.

The muon: who ordered that

What we now call the muon was initially spotted in 1936 as cloud chamber tracks with anomalously low ionisation by Seth Neddermeyer and Carl Anderson [36] and independently by J. Street and E. Stevenson [34]. The particle (figure 17c) is assumed to have the same charge as an electron, but the reduced ionisation suggested this was a new particle with a mass in between that of an electron and proton. It was a initially misidentified as the meson responsible for binding nuclei together that Hideki Yukawa predicted just a little earlier in 1935 predicted to be around 200 MeV.

Theory and experiment seemed to be lining up nicely. However, Rossi and Nereson measured the lifetime in 1943 to be 2.2 ± 0.2 μs [37], longer than that expected for Yukawa's meson. Then in 1946 at Rome, M. Conversi, E. Pancini and O. Piccioni [38] measured the absorption rate of this particle when fired at iron and carbon nuclear targets. They saw very poor

in physics went to Chadwick for the neutron.

absorption by nuclei compared to that predicted by Yukawa's mesons. Something appeared terribly inconsistent about this particle being the mediator of the strong force. Fortunately, cosmic-ray measurements by the Bristol group had identified the pion in 1947, which we shall see in section 8.

The muon is now known as the heavier cousin to the electron, possessing the same properties except its mass. Its existence is completely unexpected. It is the first particle discovered beyond the first generation, opening the **problem of flavour**. Why are there different generations of matter? We do not know. Isodor Rabi famously quipped "who ordered that?" in reacting to the muon. This remains an open question in physics today.

Contemporary cosmic-ray spectrum

For completeness, let us briefly finish the story of cosmic rays here given their astrophysical origins are less mysterious than a century ago. Figure 18b shows the contemporary fluxes [35] of various particles with altitude, dominated by muons and neutrinos before transitioning to protons above $\gtrsim 7$ km. Atmospheric neutrinos underpin the discovery of neutrino oscillations (subsection 16.2). Figure 19 shows a composition of state-of-the-art cosmic-ray energy spectra measured by experiments worldwide and in space. The flux approximately follows a steep power law

$$\frac{\mathrm{d}N}{\mathrm{d}E} \propto E^{-\gamma}, \quad \gamma \approx 2.7. \tag{2.16}$$

The vertical axis is multiplied by E^2 to improve clarity. The plot indicates the flux of cosmic ray events per area per second as diagonal bands.

Measurements today are broadly divided into two classes:

- **Space-based low energy**. At low energies $E \lesssim 10^5$ GeV, the flux is sufficiently large that space-based experiments located in orbit can directly measure the primary cosmic ray. The first dedicated space-based cosmic-ray detector was the Payload for Antimatter Matter Exploration and Light-nuclei Astrophysics (PAMELA) and an ongoing experiment is the Alpha Magnetic Spectrometer (AMS-02) on the International Space Station. Cosmic Ray Energetics and Mass (CREAM) was a high-altitude balloon detector launched in Antarctica.

  These provide high granularity measurements up to 10^5 GeV for hadrons and 10^3 GeV for electrons above which the space-based detectors are too small to accumulate sufficient statistics. The detectors possess particle identification capabilities between hadrons, electrons, positrons, and photons. They can also distinguish nuclei species

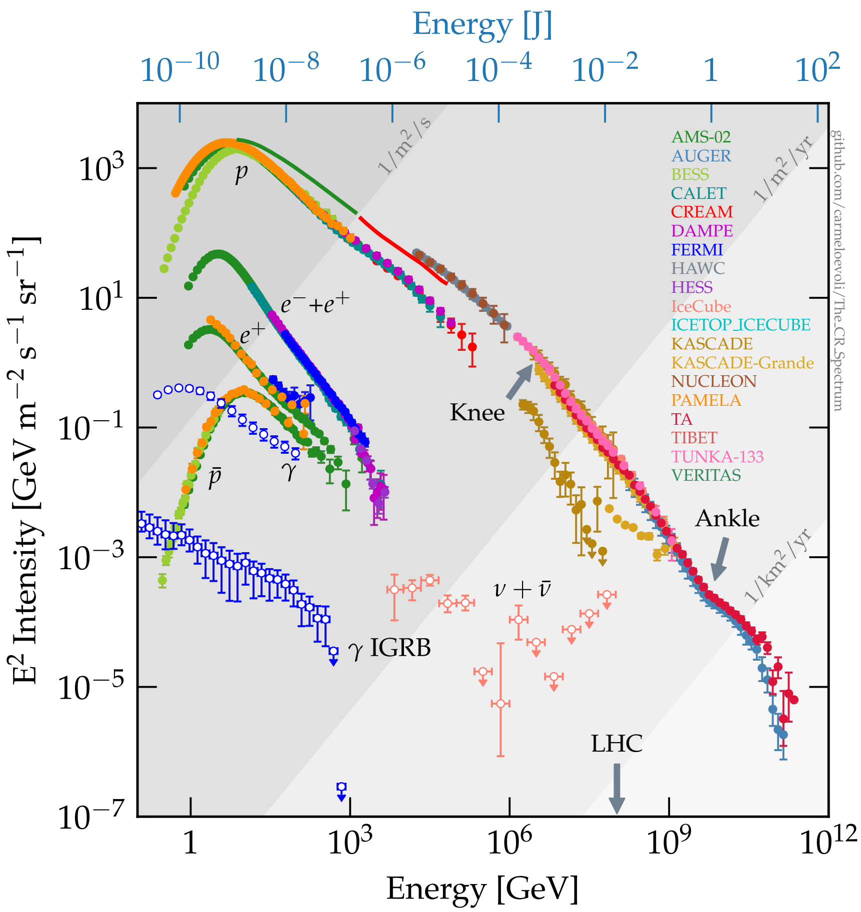


Figure 19: **Cosmic-ray energy spectrum**. The intensity is multiplied by a factor of E^2. Space-based measurements with composition identification are possible below $\lesssim 10^5$ GeV (coloured points), above which ground-based observatories are needed and only measure all-particle fluxes (black points). The main text briefly discusses the knee and ankle features. Figure from Ref. [39].

via mass spectrometry, showing that helium, oxygen to iron are not insignificant compared to protons.

- **Ground-based high energy**. At high energies $E \gtrsim 10^5$ GeV, large exposure areas are required to gather statistics at ground-based observatories. These require arrays of Cerenkov detectors covering large areas $\gtrsim 100$ km^2. However, such experiments cannot directly measure the chemical composition of the cosmic rays. They can only measure the secondary **extensive air showers** after the impact of the primary cosmic-ray particle. Non-perturbative dynamics of the strong force underpin the initial stages of the hadron interactions, requiring phenomenological models tuned to LHC data [40]. The largest of these air shower arrays include the Telescope Array in Utah, KASCADE and Pierre Auger Observatory in Argentina. They measure the atmospheric Cerenkov light induced by such showers, effectively turning the atmosphere into a calorimeter.

Figure 19 shows interesting kinks in the spectrum deviating from the $E^{-2.7}$ trend: the **knees** around 10^{15} eV and 10^{17} eV, and **ankle** just above 10^{18} eV. Cosmic rays below $\lesssim 10^{16}$ most likely originate from inside our Milky Way galaxy. A conventional interpretation of the knees arise from supernova remnants accelerating protons to a maximum energy $E_{\text{max}}^{\text{proton}} \sim 10^{15}$ eV, sometimes called pevatrons. This scales with the nuclei proton number as

$$E_{\text{max}}(Z) = Z \times E_{\text{max}}^{\text{proton}}. \tag{2.17}$$

So the knee arises from the rate of galactic protons falling off in favour of heavier nuclei up to iron, which then also falls off at the second knee. This also coincides with the critical energies needed for extragalactic cosmic rays to penetrate our galaxy's magnetic field. This gives rise to the ankle around $E \sim 5 \times 10^{19}$ eV and by studying the arrival direction, the Auger collaboration recently reported evidence supporting their extragalactic origins [41]. Finally, the suppressed flux at the most extreme energies $E \sim 10^{20}$ eV corresponds to This likely arises from protons colliding with photons from the cosmic microwave background creating an excited Delta baryon

$$p + \gamma_{\text{CMB}} \to \Delta^+ \to \begin{cases} p + \pi^0 \\ n + \pi^+ \end{cases} . \tag{2.18}$$

This is called the Greisen–Zatsepin–Kuzmin (GZK) cutoff first hypothesised in 1966.

3 Relativistic quantum mechanics

Having covered the pioneering experiments from the early twentieth century that gave birth to particle physics, we now briefly review the contemporary theoretical concepts from special relativity and quantum mechanics. This also conveniently establishes mathematical notation and units. We then make these two theories compatible, leading to the Dirac equation, the relativistic wave equation describing electron motion.

3.1 Relativity and quantum physics

We briefly review the physics of the very fast and very small. These are governed by special relativity and quantum mechanics, respectively.

Special relativity

The central tenet of special relativity is a constant speed limit c in the universe, which is the speed of light. This is defined to be exactly[31]:

$$c = 299\,792\,458 \text{ m s}^{-1} \simeq 3 \times 10^8 \text{ m s}^{-1}. \tag{3.1}$$

This is a profound statement that the speed of light is constant no matter how fast an observer is moving, and there is no absolute rest frame. The empirical foundation is the Michelson–Morley experiment in 1887 [42], and leads to the striking phenomena of time dilation and length contraction of relativity. Most particles only live briefly and let's say X has a rest lifetime τ_{rest}. Then if we run past the particle or it flies from us very quickly at speed $\beta = v/c$, we see a dilated observed lifetime:

$$\tau_{\text{observed}} = \gamma \tau_{\text{rest}} = \frac{1}{\sqrt{1-\beta^2}} \tau_{\text{rest}}, \tag{3.2}$$

where $\gamma = 1/\sqrt{1-\beta^2}$. In general, a boost along direction x at speed v results in Lorentz transformed (primed) time and space coordinates:

$$\mathsf{X}'^{\mu} = \Lambda^{\mu}_{\nu} \mathsf{X}^{\nu}, \tag{3.3}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma\beta & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{3.4}$$

[31]Since 1983 at the 17th Confŕence Générale des Poids et Mesures https://www.bipm.org/en/committees/cg/cgpm/17-1983/resolution-1

The X^μ objects are four-vectors comprising time for the $\mu = 0$ index and space components in the $\mu = 1,2,3$ indices. The Λ^μ_ν is the Lorentz transformation matrix.

When objects move at speeds approaching that of light $v \to c$, space and time are not so different as they seem at low speeds i.e. ct and x have comparable magnitudes. These two coordinates are merely related by a fancy rotation called a Lorentz boost, all due to spacetime symmetries. This is analogous to physics occurring in the up-down direction (often called y) being the same as that occurring in the left-right (often called x), related by a spatial rotation. Just as we do not measure x coordinates in miles and y coordinates in millimetres with a conversion ratio

$$x = \frac{\text{mile}}{\text{millimetre}} y. \tag{3.5}$$

It is therefore convenient to adopt notation that absorbs the scale factor into the time coordinate, which amounts to setting $c = 1$. Of course, everyday life also measures distances in units of time when there is a fixed speed e.g. "I am 30 minutes away" or "my transcontinental flight is 6 hours long".

Mass–energy equivalence Einstein also showed the mass–energy equivalence. When a particle decays, it converts its mass into the energy to create its child particles and its kinetic energy. We can construct a quantity called the rest mass m_{rest} of a particle by squaring its energy and subtracting its momentum squared:

$$(m_{\text{rest}}c^2)^2 = E^2 - (\mathbf{p}c)^2. \tag{3.6}$$

This quantity is Lorentz invariant i.e. mc^2 is the same no matter how fast we run passed the particle in what are called boosts. This fact makes Lorentz invariant quantities really interesting to physicists. We can write objects with units of momentum P^μ, with superscript μ denoting components, called the four-momentum:

$$\mathsf{P}^\mu = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}. \tag{3.7}$$

This is a vector with four components comprising the energy E/c and the three spatial components of the momentum vector of a particle. Then we define a dot product rule that all four-vectors obey upon multiplication, which has a minus sign for the spatial components:

$$\mathsf{P}^2 = \mathsf{P}^{\mathrm{T}} \cdot \mathsf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (mc)^2. \tag{3.8}$$

This operation therefore reproduces the Lorentz invariant rest mass as in eq. (3.6). If we ran passed this particle such that we (in the primed frame) see the particle with boosted four-momentum $\mathsf{P}' = (E'/c, \mathbf{p}')^{\mathrm{T}}$, we still obtain the same invariant mass squared when squaring

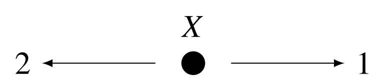
$\mathsf{P} \cdot \mathsf{P} = (mc)^2 = \mathsf{P}' \cdot \mathsf{P}'$. We can represent this useful operation of a four-vector dot product $\cdot$ with a diagonal 4×4 matrix called the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{3.9}$$

where the subscripts $\mu, \nu = \{0,1,2,3\}$ number the column and row elements of the matrix. This allows us to lower and raise indices, which imparts a negative sign on the spatial components

$$\mathsf{P}_\mu = \eta_{\mu\nu} \mathsf{P}^\mu = \begin{pmatrix} E/c \\ -\mathbf{p} \end{pmatrix}. \tag{3.10}$$

Energy and momentum conservation Another feature of the universe is energy and momentum conservation. This is actually related to the fact spacetime looks the same even if we translate it in time and space due to Noether's theorem. A simple process we can illustrate this with is a particle X decaying into particle 1 and 2 (e.g. a Higgs boson decaying to a muon and antimuon $h \to \mu^- \mu^+$):



$$\mathsf{P}_2 = \begin{pmatrix} E_{\mathbf{p}_2}/c \\ \mathbf{p}_2 \end{pmatrix} \quad \mathsf{K} = \begin{pmatrix} m_X c \\ \mathbf{0} \end{pmatrix} \quad \mathsf{P}_1 = \begin{pmatrix} E_{\mathbf{p}_1}/c \\ \mathbf{p}_1 \end{pmatrix} \tag{3.11}$$

Four-vectors make it easy to simultaneously conserve energy and linear momentum:

$$\mathsf{K} = \mathsf{P}_1 + \mathsf{P}_2. \tag{3.12}$$

We can calculate the energy spectrum of each particle by squaring (setting $c = 1$ for clarity)

$$\mathsf{P}_1^2 = (\mathsf{K} - \mathsf{P}_2)^2 = \mathsf{K}^2 + \mathsf{P}_2^2 - 2\mathsf{K} \cdot \mathsf{P}_2, \tag{3.13}$$

$$\Rightarrow m_1^2 = m_2^2 + m_X^2 - 2 m_X E_2. \tag{3.14}$$

A similar manipulation for swapped $1 \leftrightarrow 2$ labels $\mathsf{P}_2^2 = (\mathsf{K} - \mathsf{P}_1)^2$ then gives

$$E_1 = \frac{m_X^2 + m_1^2 - m_2^2}{2 m_X}, \quad E_2 = \frac{m_X^2 + m_2^2 - m_1^2}{2 m_X}. \tag{3.15}$$

This tells us that each particle in a 2-body decay has its energy uniquely determined by the masses of the three particles $X,1,2$. This is why seeing a continuous energy spectrum for electrons in beta decay provided evidence of an additional invisible particle i.e. a 3-body decay.

Electromagnetism Let us also briefly review Maxwell's equations, which in SI units look like

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0}\quad \nabla\times\mathbf{B}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}=\mathbf{j}, \tag{3.16}$$

$$\nabla\cdot\mathbf{B}=0\quad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0. \tag{3.17}$$

The upper two equations are the equations that tell us how the electromagnetic fields respond to sources of charge and current, while the lower equations relate the fields themselves.

As undergraduates learn towards the end of a course on electromagnetism, the speed of light is built into Maxwell's equations $c^2=1/\sqrt{\mu_0\varepsilon_0}$. This spectacularly unifies the naïvely disparate phenomena of wave optics, electricity and magnetism, while betraying its deep connection to relativity. In Heaviside-Lorentz units $\varepsilon_0=1$ and $\mu_0=1$ with only factors of c appearing so Maxwell's equations look like

$$\nabla\cdot\mathbf{E}=\rho\quad \nabla\times\mathbf{B}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{\mathbf{j}}{c}, \tag{3.18}$$

$$\nabla\cdot\mathbf{B}=0\quad \nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0. \tag{3.19}$$

This certainly looks symmetric especially in the absence of charges and currents $\rho=0,\mathbf{j}=\mathbf{0}$, while setting $c=1$. Recall also that vector calculus implies we can write the electromagnetic fields in terms of the scalar ϕ and vector $\mathbf{A}$ potentials defined by

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{B}=\nabla\times\mathbf{A}. \tag{3.20}$$

While classically equivalent, it turns out the potentials are important in quantum mechanical interactions with electromagnetism.

Quantum mechanics

Underpinning quantum mechanics is the **wave–particle duality**. In the nineteenth century, Young's slit experiments showed light behaves as waves, until the photoelectric effect evidence its particle nature. The Planck's law relates the energy E of a particle to its frequency f with Planck's constant h:

$$E=hf=\hbar\omega, \tag{3.21}$$

where the Planck's constant is

$$h = 6.62607015 \times 10^{-34} \text{ J s}. \tag{3.22}$$

It is often convenient to work in the angular frequencies $\omega = 2\pi f$ with the reduced Planck's constant $\hbar = h/2\pi$. Planck's constant has units of [action] = [energy] $\times$ [time] or [distance] $\times$ [momentum], which is related to the uncertainty principle at the heart of quantum physics:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}. \tag{3.23}$$

Similarly, the electron was first established as a particle in cathode ray experiments until double slit experiments illustrated its wave behaviour. The de Broglie relation relates a particle's linear momentum p_x (along the x axis) to its wavelength λ:

$$p_x = 2\pi/\lambda = \hbar k_x, \tag{3.24}$$

which generalises to a three-vector $\mathbf{p} = \hbar \mathbf{k}$ for the three spatial components.

Given we work with waves, it is often useful to talk about the frequency ω or wavenumber k. This can be formalised with the Fourier transform, where we work in a convention that arises naturally from particle in a box boundary conditions such that the inverse Fourier transform comes with factors of $\mathrm{d}^n k_i/(2\pi)^n$:

$$f(k) = \int \mathrm{d}^4 x \, \tilde{f}(x) \mathrm{e}^{\mathrm{i}k\cdot x}, \tag{3.25}$$

$$\tilde{f}(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} f(k) \mathrm{e}^{-\mathrm{i}k\cdot x}, \tag{3.26}$$

working in $n = 4$ dimensions.

Schrödinger equation In non-relativistic quantum mechanics, we can describe the an electron ψ moving freely as waves using a complex exponential to represent a wave

$$\psi = \psi_0 \mathrm{e}^{-\mathrm{i}(Et - \mathbf{p}\cdot\mathbf{x})/\hbar}. \tag{3.27}$$

This is the electron wavefunction, whose equation of motion is the Schrödinger equation. This tells us how a particle ψ of mass m moves merrily through free space with kinetic energy Hamiltonian $\hat{H} = \mathbf{p}^2/2m$:

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi. \tag{3.28}$$

We can write energy and momentum as differential operators that act on the wavefunction:

$$\hat{E} = \mathrm{i}\hbar \frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} = -\mathrm{i}\hbar \boldsymbol{\nabla}, \tag{3.29}$$

where $\boldsymbol{\nabla}$ is the three-vector of spatial derivatives. We will usually drop the hat $\hat{E} \to E, \hat{\mathbf{p}} \to \mathbf{p}$ notation for operators in the rest of this text where quantum mechanics is implied.

Particle spin Quantum mechanics also introduces us to the idea of **intrinsic spin** of a particle measured in units of angular momentum $\hbar$. A particle can be a boson (integer spin) or a fermion (half integer spin). When two particles in a system, exchange of labels their wavefunction has no sign change for bosons and a sign change for fermions

$$\psi_a(x_1)\psi_b(x_2) = \psi_b(x_1)\psi_a(x_2) \quad \text{bosons,} \tag{3.30}$$
$$\psi_a(x_1)\psi_b(x_2) = -\psi_b(x_1)\psi_a(x_2) \quad \text{fermions.} \tag{3.31}$$

Born rule The probability P of finding a particle in a particular state is given by a complex number called its quantum amplitude $\mathscr{A}$ and multiplying it by its complex conjugate, which is the Born rule:

$$P = |\mathscr{A}|^2 = \mathscr{A}^*\mathscr{A}. \tag{3.32}$$

Occasionally, we will write the wavefunction in what is called a Dirac 'ket' $|\psi\rangle$ whose complex conjugate is called a 'bra' $\langle\psi|$, so the amplitude is formed as a 'bra-ket' $\langle\psi|\psi\rangle$.

3.2 Dirac equation

A very first combination of relativity and quantum mechanics relates the Planck equation describing energy-frequency relation with the Einstein mass-energy equivalence:

$$hf = E = mc^2. \tag{3.33}$$

Using $f = c/\lambda$, this gives the (reduced) **Compton wavelength** ($\lambda\!\!\!^-$) λ_C of a particle:

$$\lambda_C = \frac{h}{mc}, \quad \lambda\!\!\!^-_C = \frac{\hbar}{mc}. \tag{3.34}$$

This is the wavelength of a particle when all its energy resides in its rest mass.

As an aside, particle physics works with fast and small things so it is convenient to adopt a system of units that absorb factors of $\hbar$ and c into variables like energy and distance. We will often work in **natural units** $\hbar = 1, c = 1$, where a very helpful quantity to convert energy into distance quantities is:

$$\hbar c \simeq 197\ \text{MeV fm}. \tag{3.35}$$

This tells us a particle with an energy scale around 200 MeV energy has a wavelength of around a femtometre.

Klein–Gordon equation

Now we desire a description of quantum mechanical particles at relativistic speeds beyond just the rest mass wavelength λ_C. Specifically, we seek a quantum mechanical equation of motion that respects Lorentz invariants $E^2 - (pc)^2 = (mc^2)^2$ rather than the non-relativistic kinetic energy $p^2/2m$. The simplest way is to substitute the quantum mechanical energy-momentum operators of equation (3.29) into (3.6) to obtain:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2\right)\phi = \left(\frac{mc}{\hbar}\right)^2\phi, \tag{3.36}$$

such that $\phi = \mathrm{e}^{-\mathrm{i}(Et - \mathbf{p}\cdot\mathbf{x})}$ is a free-space solution that satisfies the Lorentz invariant requirement. This is the **Klein–Gordon equation**. We now know this can describe the relativistic motion of scalar fields. But this was historically set aside due to confusion around negative energy states and negative probability currents. As a notational aside, you will sometimes see the shorthand $\Box$ for the d'Alembert wave operator

$$\Box = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2, \tag{3.37}$$

which allows us write the Klein-Gordon equation (3.36) in a compact quadratic form (setting $\hbar = c = 1$ for visual clarity)

$$(\Box + m^2)\phi = 0. \tag{3.38}$$

This uses the definition of the four-gradient ∂_μ, which is a four-vector of time and space derivatives

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{\partial}{\partial t} \\ +\boldsymbol{\nabla} \end{pmatrix}, \quad \partial^\mu = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{\partial}{\partial t} \\ -\boldsymbol{\nabla} \end{pmatrix}. \tag{3.39}$$

Constructing the Dirac equation

Historically, Paul Dirac instead sought to write an equation that was first order in time. This therefore mimics the Schrödinger equation where only the initial position (not the velocity) is required as the initial condition. The goal is to "take the square root" of the Klein–Gordon operator by somehow "completing the square", heuristically

$$\Box + m^2 \stackrel{?}{=} (\sqrt{\Box} + \mathrm{i}m)(\sqrt{\Box} - \mathrm{i}m). \tag{3.40}$$

Needless to say, "$\sqrt{\Box}$" is mathematically ill-defined: what exactly does it mean to take the square root of a second order differential operator? Undeterred, Dirac pressed ahead and postulated an equation having only first derivatives in time and space with unknown $A, \mathbf{B}$ coefficients:

$$\left(A\frac{\partial}{\partial t} + \mathbf{B}\cdot\boldsymbol{\nabla}\right)\psi = m\psi. \tag{3.41}$$

For convenient algebraic manipulation, we define a shorthand for spacetime derivatives:

$$\partial_t \equiv \frac{\partial}{\partial t}, \quad B_i\partial_i \equiv \sum_{i=1}^{3} B_i\partial_i = B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z} = \mathbf{B}\cdot\boldsymbol{\nabla}. \tag{3.42}$$

Focusing on the left-hand side of equation (3.41), we can square the equation by multiplying through by the differential operators then expanding to yield:

$$(A\partial_t + B_i\partial_i)^2 = [A^2\partial_t^2 + \underbrace{A\partial_t(B_i\partial_i) + (B_i\partial_i)A\partial_t}_{=(AB_i+B_iA)\partial_i\partial_t} + B_i\partial_i B_j\partial_j]. \tag{3.43}$$

We can factorise the cross-terms shown in the underbrace because (i) time and spatial partial derivatives commute $\partial_i\partial_t = \partial_t\partial_i$, and (ii) A, B_i have neither time nor space dependence so also commute with the ∂_t and ∂_i derivatives. Therefore, equation (3.41) becomes

$$[A^2\partial_t^2 + (AB_i + B_iA)\partial_i\partial_t + B_iB_j\partial_i\partial_j]\psi = m^2\psi. \tag{3.44}$$

For this to match the Klein-Gordon equation (3.36), Dirac required $A^2 = -1$ and all cross-terms in derivatives like $\partial_t\partial_i, \partial_x\partial_y$ to vanish:

$$A^2 = -1, \quad AB_i + B_iA = 0, \quad B_iB_j = \delta_{ij}, \tag{3.45}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. For A and B_i to satisfy these intriguing relations, Dirac astutely realised that they must be matrices. With some notational foresight, we can define $A = \mathrm{i}\gamma^0, B_i = \mathrm{i}\gamma_i$ in terms of objects we now call Dirac matrices γ_μ, which satisfy

$$(\gamma_0)^2 = 1, \quad \gamma_0\gamma_i + \gamma_i\gamma_0 = 0, \quad \gamma_i\gamma_j = -\delta_{ij}. \tag{3.46}$$

This shows that time and spatial components multiply to give a relative negative sign while all the off-diagonal elements vanish, which sounds awfully familiar. Indeed, this tells us how swapping γ_μ gives a minus sign plus an extra bit proportional to the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ (3.9):

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}. \tag{3.47}$$

This is an **anti-commutation relation**, where the curly braces define an anti-commutator $\{a, b\} \equiv ab + ba$. we call this the Clifford algebra, after William Clifford who first studied these anti-commuting mathematical objects in 1878. This anti-commutation is a defining feature of fermions. Substituting $A = \mathrm{i}\gamma^0, B^i = \mathrm{i}\gamma^i$ back into equation (3.41), this becomes the celebrated **Dirac equation** first written[32] in 1928

$$\boxed{\mathrm{i}\gamma^\mu \partial_\mu \psi = m\psi.} \tag{3.48}$$

This astonishing equation describes the quantum mechanical and relativistic motion of spin-half particles in free space. It also shows how anti-commuting gamma matrices γ^μ are a mathematical consequence in trying to define this operator $\sqrt{\Box} = \gamma^\mu \partial_\mu$ to make quantum mechanics consistent with special relativity.

3.3 Spinor and antimatter waves

We now explore the remarkable anatomy of the Dirac equation, first antimatter and spinors that underpin particle physics.

Weyl representation of Dirac equation

One way to write the solutions of the Dirac equation is to work in a specific representation that satisfies the defining Clifford algebra (3.47). The Weyl (chiral) representation is

$$\gamma^\mu = \begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbf{0} \end{pmatrix}, \quad \text{where} \quad \sigma^\mu = \begin{pmatrix} \mathbb{I} \\ \boldsymbol{\sigma} \end{pmatrix}, \quad \bar{\sigma}^\mu = \begin{pmatrix} \mathbb{I} \\ -\boldsymbol{\sigma} \end{pmatrix}, \tag{3.49}$$

with $\mathbb{I}$ being the 2-by-2 identity matrix and $\boldsymbol{\sigma}$ is the 3-vector of Pauli matrices familiar from quantum mechanics:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}, \quad \text{where} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.50}$$

The Lorentz index μ on γ^μ makes these objects look like a four-vector, but they are not i.e. they do *not* transform as $\gamma^\mu \to \Lambda^\mu_\nu \gamma^\nu$ because they remain invariant. When we write out the components, we can see explicitly that they are just a collection of matrices with complex numbers; the Lorentz index serves as a mnemonic for which element of a four-vector they multiply with.

[32] As it appears on a plaque situated in Westminster Abbey, London.

Equipped with this explicit representation of matrices (3.49) and $E = \mathrm{i}\partial_t, \mathbf{p} = -\mathrm{i}\boldsymbol{\nabla}$, we can write the Dirac equation (3.48) explicitly

$$\mathrm{i}\begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix} \psi = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \psi \tag{3.51}$$

In the chiral representation, the **Dirac spinor** ψ is a four-component spinor made by stacking a pair of two-component Weyl spinors ϕ, χ (3.82)

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi_a \\ \phi_b \\ \chi_a \\ \chi_b \end{pmatrix}, \tag{3.52}$$

Writing out (3.51) gives the pair of coupled differential equations mixing the upper and lower two-spinors

$$\mathrm{i}\sigma^\mu \partial_\mu \chi = m\phi, \tag{3.53}$$

$$\mathrm{i}\bar{\sigma}^\mu \partial_\mu \phi = m\chi. \tag{3.54}$$

It will turn out that ϕ corresponds to right-handed particles and χ with left-handed particles in the massless limit, as we shall now see. Sometimes you will see this written as $\psi_R = \phi, \psi_L = \chi$.

Weyl equations and helicity

As usual in physics, it is insightful to take some limits to study its consequences. Something interesting happens in the ultra-relativistic limit $m \to 0$: the Dirac equation decouples into two independent **Weyl equations**:

$$\mathrm{i}\sigma \cdot \partial \phi = 0, \quad \text{"unbarred"} \tag{3.55}$$

$$\mathrm{i}\bar{\sigma} \cdot \partial \chi = 0, \quad \text{"barred"}. \tag{3.56}$$

Here we can refer to the two distinct equations as whether σ is "unbarred" and "barred" $\bar{\sigma}$. While mathematically undramatic, this decoupling is telling us something deep about nature: there are two types of massless fermions with a distinguishable property. It turns out this will correspond to left-handed or right-handed Weyl spinors. This difference in handedness is already implicit in equation (3.49) by the negative sign prefixing Pauli matrices in $\bar{\sigma}^\mu$ hinting this somehow distinguishes rotations. In the $m \to 0$ limit, they never mix: a left-handed Weyl fermion stays left-handed; a right-handed Weyl fermion stays left-handed.

We can solve the two Weyl equations using a plane wave ansatz

$$\phi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} \mathrm{e}^{-\mathrm{i}(Et-\mathbf{p}\cdot\mathbf{x})}, \quad \chi = \begin{pmatrix} \chi_a \\ \chi_b \end{pmatrix} \mathrm{e}^{-\mathrm{i}(Et-\mathbf{p}\cdot\mathbf{x})}. \tag{3.57}$$

Because the 2×2 Pauli matrices are built in, the solutions are necessarily two components. The Weyl equations then become

$$(\sigma\cdot\mathsf{P})\,\phi = 0, \quad \Rightarrow \quad (\mathbb{I}E - \boldsymbol{\sigma}\cdot\mathbf{p})\phi = 0, \tag{3.58}$$

$$(\bar{\sigma}\cdot\mathsf{P})\,\chi = 0, \quad \Rightarrow \quad (\mathbb{I}E + \boldsymbol{\sigma}\cdot\mathbf{p})\chi = 0. \tag{3.59}$$

Focusing on the unbarred Weyl equation (3.58), we can align the momentum along the z-axis $p_z = |\mathbf{p}|$ to find this eigenvalue equation

$$\mathbb{I}E - \sigma_z|\mathbf{p}| = \begin{pmatrix} E-|\mathbf{p}| & 0 \\ 0 & E+|\mathbf{p}| \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = 0. \tag{3.60}$$

This eigenvalue problem has two solutions with positive and negative energies, which we interpret as particles and anti-particles:

$$\phi_+ = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \quad E = +|\mathbf{p}|, \quad \text{particle}, \tag{3.61}$$

$$\phi_- = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \quad E = -|\mathbf{p}|, \quad \text{anti-particle}. \tag{3.62}$$

The unbarred Weyl equation (3.55) becomes

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\phi = \frac{E}{|\mathbf{p}|}\phi, \tag{3.63}$$

We call this operator $\hat{h} = \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}/|\mathbf{p}|$ the **helicity**. This is the projection of the spin along the direction of the momentum $\mathbf{p}$. In equation (3.51), the Pauli operator $\boldsymbol{\sigma}$ generates a sense of rotation projected onto linear momentum vector (recall $\mathbf{p} = -\mathrm{i}\boldsymbol{\nabla}$) that defines the axis-of-rotation direction. The eigenstates with well-defined helicity eigenvalues h are:

$$\mathrm{i}\sigma\cdot\partial\phi = 0 \;\Rightarrow\; \begin{cases} \phi_+ = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), & h = +1, \quad E > 0, \quad \text{right-handed particle}, \\ \phi_- = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), & h = -1, \quad E < 0, \quad \text{left-handed anti-particle}. \end{cases} \tag{3.64}$$

This is the physical interpretation of the unbarred Weyl equation (3.55): it is the equation of motion for a right-handed $h = +1$ particle $E > 0$ and left-handed $h = -1$ anti-particle $E < 0$.

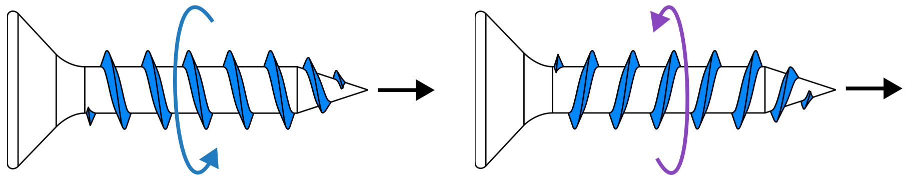


Figure 20: **Left- and right-handed threaded screw**. I constructed the left-handed screw from the original right-handed screw by performing a parity transformation (vertical flip) in my graphics software. Figures: adapted from Wikimedia

.

We determine their handedness by whether the spin operator is aligned or anti-aligned with momentum:

$$\text{RH}: \quad \underset{\mathbf{s}}{\Rightarrow} \cdot \underset{\mathbf{p}}{\rightarrow} = + = \text{aligned}, \tag{3.65}$$

$$\text{LH}: \quad \underset{\mathbf{s}}{\Rightarrow} \cdot \underset{\mathbf{p}}{\leftarrow} = - = \text{anti-aligned}. \tag{3.66}$$

So ultra-relativistic spinors are like threaded screws. Their handedness arises from a sense of rotation in the x-y plane being aligned or anti-aligned with its z-axis of rotation (figure 20).

You can repeat this exercise with the barred Weyl equation (3.59) to find the two solutions $\chi_\pm$ correspond to:

$$\mathrm{i}\bar{\sigma}\cdot\partial\chi = 0 \;\Rightarrow\; \begin{cases} \chi_+ = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), & h=+1, \quad E<0, \quad \text{right-handed anti-particle}, \\ \chi_- = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), & h=-1, \quad E>0, \quad \text{left-handed particle}. \end{cases} \tag{3.67}$$

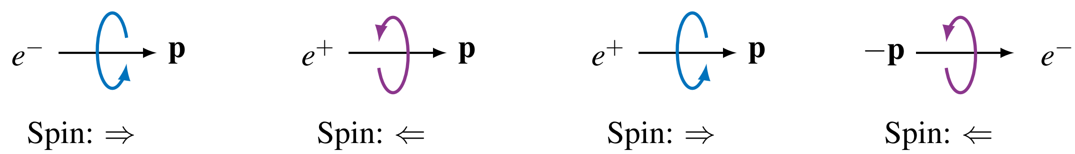
Figure 21 summarises these four possible eigenstates of the Weyl equations. The literature often uses the following u, v notation to denote particles and antiparticles:

$$u_{\mathrm{R}} = \text{right-handed particle} \qquad v_{\mathrm{R}} = \text{right-handed antiparticle}, \tag{3.68}$$

$$u_{\mathrm{L}} = \text{left-handed particle} \qquad v_{\mathrm{L}} = \text{left-handed antiparticle}. \tag{3.69}$$

All these sketches make it look like the two Weyl equations are mirror images of each other. Indeed your left and right hands exchange when looking at a mirror in everyday life, which we mathematically formalise as a parity P transformation $\mathbf{r} \to -\mathbf{r}$ exchanging the handedness of Weyl spinors

$$P: \psi_{L,R} \to \psi_{R,L}, \quad \mathbf{r} \to -\mathbf{r}. \tag{3.70}$$



(a) RH particle $\boldsymbol{\sigma}\cdot\mathbf{p} \propto +$ (RH) Particle $E > 0$

(b) LH anti-particle $\boldsymbol{\sigma}\cdot\mathbf{p} \propto -$ (LH) Anti-particle $E < 0$

(c) RH anti-particle $\boldsymbol{\sigma}\cdot\mathbf{p} \propto +$ (RH) Anti-particle $E < 0$

(d) LH particle $\boldsymbol{\sigma}\cdot\mathbf{p} \propto -$ (LH) Particle $E > 0$

Figure 21: **Handedness of massless particle and anti-particle spinors**. The massless limit of the Dirac equations decouples into the two Weyl equations. Solutions can be LH or RH assigned to particle $E > 0$ or anti-particle $E < 0$. (a) and (b) are solutions to the unbarred Weyl equation $\mathrm{i}\sigma\cdot\partial\phi = 0$, while (c) and (d) are solutions to the barred Weyl equation $\mathrm{i}\bar{\sigma}\cdot\partial\chi = 0$.

Dirac mass and chirality

When restoring $m \neq 0$, the Weyl equations become coupled (3.54), mixing the left- and right-handed Weyl spinors ϕ, χ with well-defined helicities. This becomes the notion of **chirality**. Given further spoilers that the electroweak interactions couple only to left-handed Weyl spinors, it is convenient to define an operator to project out the left and right Weyl spinors from a Dirac spinor:

$$\phi = \psi_L = P_R\psi = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}\psi, \quad \chi = \psi_R = P_L\psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}\psi. \tag{3.71}$$

Working in this Weyl representation, we can construct a matrix with a curious "5" in its name

$$\gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{3.72}$$

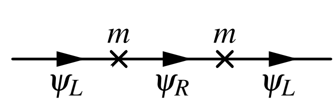
This allows us to construct **chiral projection operators** $P_{L,R}$ such that they work as (3.71), extracting the upper and lower parts of the Dirac spinor:

$$P_L = \frac{1}{2}(I + \gamma^5), \quad P_R = \frac{1}{2}(I - \gamma^5). \tag{3.73}$$

This curiously named object γ^5 satisfies these defining properties

$$\gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \quad \text{satisfying} \quad \{\gamma^\mu, \gamma^5\} = 0, \quad (\gamma^5)^2 = I. \tag{3.74}$$

The physical picture of a Dirac mass are the left and right chirality states oscillating into each other:



(3.75)

The rate of sloshing back and forth from left to right chirality is set by the mass m. We can also define the **Dirac adjoint** to be

$$\bar{\psi} = \psi^\dagger \gamma^0 = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}. \tag{3.76}$$

And a Dirac mass term looks like $m\bar{\psi}\psi$. Foreshadowing electroweak theory, the non-zero expectation value of the Higgs vacuum dynamically induces this coupling we associate with mass between left and right spinors. The Dirac equation profoundly shifts our interpretation of what mass is: this perpetual change of fermion handedness gives rise to inertial mass for Dirac particles.

Massive antimatter

We describe the negative energy states correspond to anti-particles in the massless limit as solutions to the Weyl equations. The existence of anti-particles persists for $m \neq 0$. Restoring the mass and multiplying out the Weyl equations gives

$$(\mathbb{I}E - \boldsymbol{\sigma}\cdot\mathbf{p})\chi = m\phi, \tag{3.77}$$

$$(\mathbb{I}E + \boldsymbol{\sigma}\cdot\mathbf{p})\phi = m\chi. \tag{3.78}$$

As ϕ, χ are eigenstates of energy and momentum, we can eliminate ϕ in the first equation to give

$$(\mathbb{I}E - \boldsymbol{\sigma}\cdot\mathbf{p})(\mathbb{I}E + \boldsymbol{\sigma}\cdot\mathbf{p})\chi = m^2\chi \tag{3.79}$$

Using the Pauli identity $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ijk}\sigma^k$, this reduces to the relativistic dispersion having positive and negative energy solutions

$$E_\pm(\mathbf{p}) = \pm\sqrt{|\mathbf{p}|^2 + m^2}. \tag{3.80}$$

Upon quantisation of the spinor waves, the negative energy states are associated with **antimatter**. When Dirac first wrote his equation, the full profound consequences of it were not fully appreciated. Historically, the negative energy states remained confusing. We shall skip the discussion on the "Dirac sea" picture often seen in textbooks. Instead, we are assured that once coupled to electromagnetism, these states correspond to electric **charge conjugation**. These are positrons e^+.

Antimatter explicitly allows non-conservation of particle number. This is a striking distinction to what we learn in chemistry, where conservation of particle number implies we cannot create or destroy carbon or oxygen atoms in a reaction because they merely rearrange. But the existence of antimatter upends this picture and is a central to relativistic quantum theories.

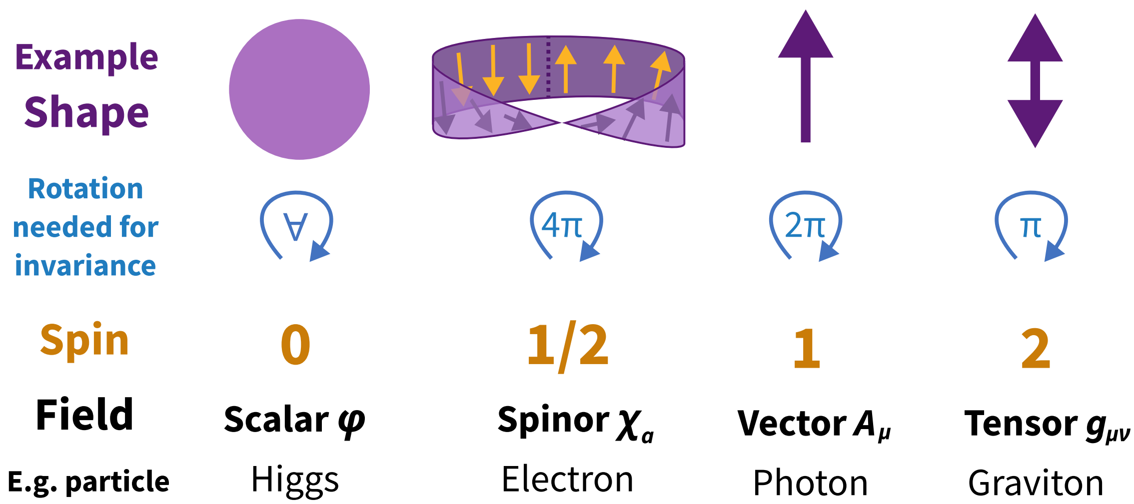


Figure 22: **Spin and rotation memory aid**. Examples of how to interpret spins as rotation with fields and particles. Spin-0 is the same for all angles, spin-1 is a single-ended arrow needing a full 2π rotation, and spin-2 is a double-ended arrow requiring a half π rotation to restore invariance. Spin half means one must perform a 4π rotation for an object to return to its original configuration: transporting an arrow round a Möbius strip is a classic example of this, where a 2π rotation results in an $\uparrow$ arrow being inverted $\downarrow$ due to the twist. I have yet to find a shape is invariant under a $4\pi/3$ rotation for a spin-3/2 particle.

Spinor rotation

Under Lorentz transformations Λ, Weyl spinors transform as

$$\psi_L \to \Lambda\psi_L = \exp\left[\frac{\mathrm{i}\sigma_i}{2}(\theta_i - \beta_i)\right]\psi_L \tag{3.81}$$

$$\psi_R \to \Lambda\psi_L = \exp\left[\frac{\mathrm{i}\sigma_i}{2}(\theta_i + \beta_i)\right]\psi_R \tag{3.82}$$

Note well the Pauli matrices σ_i, factor of $1/2$ and $\mp$ signs for boosts β_i and rotations θ_i. The physical picture of spin-1/2 particles is that a 2π rotation induces a negative sign, and only a 4π rotation restores the original configuration. You can visualise this in everyday life by twisting your arm while holding a plate of food, and following an single-ended arrowhead embedded on a Möbius strip. The twist means the arrow direction is inverted after a 2π transport that is only restored after 4π. This is contrasted with objects of other spin as visualised in figure 22.

We can write some explicit spinor solutions by working in polar coordinates

$$\mathbf{p} = E \begin{pmatrix} \sin\theta\,\cos\varphi \\ \sin\theta\,\sin\varphi \\ \cos\theta \end{pmatrix}, \tag{3.83}$$

we can expand out the right-handed Weyl equation with the aid of the Euler identity $e^{i\varphi} = \cos\varphi + i\sin\varphi$ to write all the components of the eigenvalue equation explicitly:

$$(\sigma\cdot\mathsf{P})\,u_{\mathrm{R}} = (I\mathsf{P}_0^{\pm} - \boldsymbol{\sigma}\cdot\mathbf{p}) = E\begin{pmatrix} \pm1-\cos\theta & -e^{-i\varphi}\sin\theta \\ -e^{i\varphi}\sin\theta & \pm1+\cos\theta \end{pmatrix} u_{\mathrm{R}}^{\pm} = 0, \tag{3.84}$$

where $\mathsf{P}_0^{+} = +E$ for particles (positive energy state) and $\mathsf{P}_0^{-} = -E$ for antiparticles (negative energy state). Solve this eigenvalue problem using your favourite linear algebra techniques (via pen in hand or symbolic algebra program) and apply trigonometric half-angle identities $\sin 2\theta = 2\sin\theta\cos\theta$ to obtain the normalised[33] eigenvector spinor solutions

$$u_{\mathrm{R}} \equiv u_{\mathrm{R}}^{+} = \sqrt{2E}\begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{pmatrix}, \quad v_{\mathrm{R}} \equiv u_{R}^{-} = \sqrt{2E}\begin{pmatrix} \sin(\theta/2) \\ -e^{i\varphi}\cos(\theta/2) \end{pmatrix}. \tag{3.85}$$

Solve the left-handed Weyl equation $(\bar{\sigma}\cdot\mathsf{P})u_{\mathrm{L}} = 0$ analogously to find:

$$u_{\mathrm{R}} = v_{\mathrm{L}} = \sqrt{2E}\begin{pmatrix} e^{-i\varphi/2}\cos(\theta/2) \\ e^{+i\varphi/2}\sin(\theta/2) \end{pmatrix}, \quad u_{\mathrm{L}} = v_{\mathrm{R}} = \sqrt{2E}\begin{pmatrix} e^{-i\varphi/2}\sin(\theta/2) \\ -e^{+i\varphi/2}\cos(\theta/2) \end{pmatrix}. \tag{3.86}$$

The spinors are equivalent up to any global phase, and we follow a convention that makes the complex phase explicit on both components by multiplying through by $e^{-i\varphi/2}$.

3.4 Gyromagnetic factor

Having studied the ultra-relativistic limit $E \gg m$ decouples the Dirac equation, we now turn to a key result of the non-relativistic limit. Historically, this revealed the remarkable corollary of the Dirac equation that it requires the electron intrinsic gyromagnetic factor to be $g_e = 2$. We initially approach the non-relativistic regime of the Dirac equation by taking the large mass $E \approx m$ limit $\psi_{L,R} = \chi_{L,R}e^{-imt}$

$$\begin{pmatrix} -m & I\hat{E} - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \\ I\hat{E} + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & -m \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} e^{-imt} = 0, \tag{3.87}$$

[33]The normalisation $u^{\dagger}u = v^{\dagger}v = 2E$ is conventional such that it transforms as a probability density, which is the zeroth component of the conserved four-vector current $j_{\mu} = (\rho, \mathbf{j})^{\mathrm{T}}$.

where we momentarily restore the circumflex $\hat{E}, \hat{\mathbf{p}}$ to remind ourselves that they are operators. Recognising the energy differential operator $E = \mathrm{i}\partial_t$, we use the product rule to write

$$E(\chi_{L,R}\mathrm{e}^{-\mathrm{i}mt}) = \mathrm{e}^{-\mathrm{i}mt}(m+E)\chi_{L,R}. \tag{3.88}$$

Then the Dirac equation decouples into a pair of simultaneous equations

$$-m\chi_L + (m+E-\boldsymbol{\sigma}\cdot\mathbf{p})\chi_R = 0 \tag{3.89}$$

$$-m\chi_R + (m+E+\boldsymbol{\sigma}\cdot\mathbf{p})\chi_L = 0 \tag{3.90}$$

Eliminating χ_R allows us to write

$$E + \frac{E^2}{2m} = \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}{2m} \tag{3.91}$$

As we are in the non-relativistic limit, we neglect the E^2 term quadratic in energy and recognising $E = \mathrm{i}\partial_t$, this is in the form of the Schrödinger equation. This allows us to recognise the non-relativistic limit of the Dirac equation recovers the Pauli Hamiltonian

$$H_{\text{Pauli}} = \frac{(\boldsymbol{\sigma}\cdot\mathbf{p})^2}{2m}. \tag{3.92}$$

The minimal coupling prescription to describe particles with charge q experiencing electromagnetism is the replacement $\mathbf{p} \to \mathbf{p} - q\mathbf{A}$

$$H_{\text{Pauli-EM}} = \frac{[\boldsymbol{\sigma}\cdot(\mathbf{p}-q\mathbf{A})]^2}{2m}. \tag{3.93}$$

Now for some vector calculus gymnastics to isolate the spin-magnetic coupling. First invoke the Pauli matrix identity[34] $(\boldsymbol{\sigma}\cdot\mathbf{x})(\boldsymbol{\sigma}\cdot\mathbf{y}) = \mathbf{x}\cdot\mathbf{y} + \mathrm{i}\boldsymbol{\sigma}\cdot(\mathbf{x}\times\mathbf{y})$ and momentum as a differential operator $\mathbf{p} = -\mathrm{i}\boldsymbol{\nabla}$ acting on a generic function $f(x)$

$$[\boldsymbol{\sigma}\cdot(\mathbf{p}-q\mathbf{A})]^2 f = (\mathbf{p}-q\mathbf{A})^2 f - \mathrm{i}\boldsymbol{\sigma}\cdot[(\mathrm{i}\boldsymbol{\nabla}+q\mathbf{A})\times(\mathrm{i}\boldsymbol{\nabla}+q\mathbf{A})]f. \tag{3.94}$$

Multiplying out the vector cross product, we find a non-vanishing cross term

$$(\mathrm{i}\boldsymbol{\nabla}+q\mathbf{A})\times(\mathrm{i}\boldsymbol{\nabla}+q\mathbf{A}) = \mathrm{i}q[\boldsymbol{\nabla}\times(\mathbf{A}f) + \mathbf{A}\times(\boldsymbol{\nabla}f)] + [\underbrace{\mathrm{i}^2\boldsymbol{\nabla}\times\boldsymbol{\nabla} + q^2\mathbf{A}\times\mathbf{A}}_{=0}]f \tag{3.95}$$

$$= \mathrm{i}q[f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}f) + \mathbf{A}\times(\boldsymbol{\nabla}f)], \tag{3.96}$$

[34]Derive this by working in index notation for the commutation and anti-commutation algebra of the Pauli matrices $[\sigma_i, \sigma_j] = 2\mathrm{i}\varepsilon_{ijk}\sigma_k$, $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, and vector product $(\mathbf{x}\times\mathbf{y})_i = \varepsilon_{ijk}x_j y_k$.

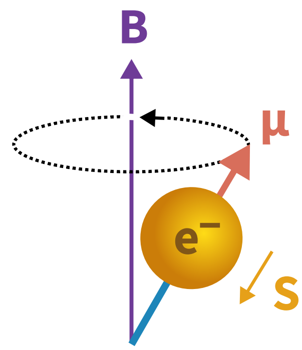


Figure 23: **Magnetic moment of intrinsic spin**. An electron with spin $\mathbf{S} = \boldsymbol{\sigma}/2$ has a magnetic moment $\boldsymbol{\mu}$, which undergoes precession in an external magnetic field $\mathbf{B}$.

recalling the antisymmetry of the cross-product yields a negative sign on applying the differentiation product rule. Identifying $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, the Pauli Hamiltonian simplifies to

$$H_{\text{Pauli-EM}} = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} - \frac{q}{2m}\boldsymbol{\sigma} \cdot \mathbf{B}. \tag{3.97}$$

Compare this with the general Hamiltonian for an intrinsic magnetic dipole moment $\boldsymbol{\mu} = (g_s q/2m)\mathbf{S}$ interacting with an external magnetic field $\mathbf{B}$

$$H_{\text{magnetic}} = -\boldsymbol{\mu} \cdot \mathbf{B} = \left(\frac{g_s q}{2m}\mathbf{S}\right) \cdot \mathbf{B} = -\left(\frac{g_s q}{2m}\frac{\boldsymbol{\sigma}}{2}\right) \cdot \mathbf{B}, \tag{3.98}$$

having used the definition $\mathbf{S} = \boldsymbol{\sigma}/2$. Figure 23 shows this spin-magnetic precession of an electron, which is usually studied in non-relativistic quantum mechanics classes. For H_{magnetic} to agree with the Pauli-EM Hamiltonian (3.97), we find the spin gyromagnetic factor g_s must be

$$\boxed{g_s = 2.} \tag{3.99}$$

Historically, this was a landmark moment in relativistic quantum theory. In Pauli theory, g_s is a free parameter relating $\boldsymbol{\mu}$ with $\boldsymbol{\sigma}$, whose value of two is determined experimentally. By contrast, Dirac theory explains this must be two by the consistent union of quantum mechanics with special relativity.

To summarise, the Dirac equation yields several profound results:

- Mass is an oscillation between left-handed and right-handed fermions.
- Unexpected prediction of antimatter, implying particle number is not conserved.
- Theoretical basis for Pauli's phenomenological description of spin.

- Minimal coupling to electromagnetism implies intrinsic gyromagnetic factor of two.

Together, these developments pave the foundations for quantum electrodynamics. While initially applied to the electron, the central importance of the Dirac equation is that it holds for all fundamental spin-1/2 particles carrying electric charge. This therefore underpins the description of charged leptons and quarks in the Standard Model.