Homework assignments

NYU PHYS-UA 123: Quantum Mechanics I
Fall 2026 Semester
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Submission deadlines: by 11:59pm of due date at top of each homework on Brightspace.

HW1: Units and wave-particle duality (set Sep 3, due Sep 10)

1. Planck's constant $h = 6.626 \times 10^{-34}$ J s, speed of light $c = 2.9979 \times 10^8$ m s^{-1}, and 1 eV $= 1.602 \times 10^{-19}$ J. Find the numerical value of hc and $\hbar c$ in (a) Joule-metres J m and (b) show that $hc = 1240$ eV nm in atomic units of electronvolt-nanometers eV nm.

2. Sketch a diagram for and describe the setup of the Davisson-Germer experiment.

   (a) Optional field trip: visit the site where this experiment occurred and find figure 1.

   (b) How does this experiment show the wave behaviour of electrons? What is meant by wave-particle duality? What is the de Broglie hypothesis and its formula?

   (c) Derive the expression for the momentum p of an electron with mass m and charge Q accelerated through a potential V. Show that the de Broglie wavelength of an electron accelerated through a voltage V is $\lambda \simeq \frac{1.23 \text{ nm}}{\sqrt{V}}$.

   (d) What is the numerical value of the voltage required such that the electron de Broglie wavelength is 100 times smaller than (i) a virus with size 20 nm and (ii) the mean diameter of a silicon atom of 0.2 nm?

   (e) Describe the quantities in the Bragg formula $d \sin\phi = n\lambda$. Dialling the incident electron beam energy of their experiment, Davisson and Germer also found an $n = 1$ peak at $\phi = 44^\circ$ for monocrystalline nickel with lattice spacing $d = 0.215$ nm. What is the de Broglie wavelength and energy $E = p^2/2m$ of the incident electron beam? Express your answers in units of eV and nm.

3. Using $E = mc^2$, show that the electron mass $m_e = 9.11 \times 10^{-31}$ kg is 511 keV/c^2.

   (a) Convert the following rest masses into eV/c^2 units:
   (i) proton $m_p = 1.67 \times 10^{-27}$ kg, (ii) onion cell 10^{-15} kg, (iii) cat $m_{\text{cat}} = 5$ kg.

   (b) Show that de Broglie wavelength is $\lambda_p = (hc)/(0.01mc^2)$ for a particle with mass m and speed $v = 0.01c$. For the electron and objects (i)–(iii), estimate numerical values for the Compton wavelength λ_C and λ_p for $v = 0.01c$.

4. Describe the Compton experiment and write down the Compton scattering formula.

   (a) Taking the incident wavelength as $\lambda = 0.01$ nm, what are the outgoing scattered wavelengths λ'_θ for scattering angles $\theta = 0^\circ, 90^\circ, 180^\circ$ in the cases of: (i) classical electromagnetism and (ii) treating light as a particle?

   (b) How does this experiment and its results demonstrate the particle nature of light?

HW2: Atoms and wavefunctions (set Sep 10, due Sep 17)

1. Describe the Frank-Hertz experiment and how the results demonstrate quantized energy levels of atoms. Consider the Bohr model of hydrogen:

   (a) What is the empirical evidence for the existence of an atomic nucleus?

   (b) By equating centripetal and electrostatic forces then using Bohr quantization $L = n\hbar$ of angular momentum $L = mvr$, show that the possible radii are $r_n = (\hbar n^2)/(mc\alpha)$ and find the expression for the dimensionless constant α.

   (c) Find an expression for the electron velocity v with circular Bohr orbits in terms of the speed of light c and quantum number n. Evaluate v/c for $n = 1$.

   (d) Evaluate the numerical pre-factors to show the energies are $E_n = -13.6\ \text{eV}/n^2$. What is the energy and wavelength of the photon for $n = 3$ transitioning to $n = 2$?

2. State what a wavefunction $\Psi(x)$ physically represents.

   (a) Using the normalization condition, what are the S. I. units of the wavefunction $\Psi(x)$ as a function of one-dimensional position x? Explain how you arrived at this answer. How does this change in three spatial dimensions?

   (b) Given a wavefunction $\Psi(x)$, what is the expression for the probability of finding a particle in $a < x < b$? What mathematical properties must $\Psi(x)$ satisfy to be considered a probability amplitude in quantum mechanics?

   (c) Given a wavefunction at $t = 0$ is $\Psi(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-a)^2}$, where $a = 1$ m, write down the integral representing the probability $P(x)$ of finding a particle between position $x = 0.3$ m and $x = 1.2$ m and sketch the area that is represented by this integral. You do not need to calculate the integral.

3. The double-slit experiment has a source, barrier with slits of width a, and screen.

   (a) Qualitatively sketch and describe the distribution on the screen for: (i) classical marbles, (ii) classical water waves with wavelength $\lambda \approx a$.

   (b) Repeat these sketches for N quantum electrons with de Broglie wavelength $\lambda \approx a$ for (iii) $N = 1$ and (iv) $N \simeq 30$.

   (c) Discuss qualitatively the similarities and differences between the classical and quantum cases. Why are classical probabilities insufficient for quantum physics?

HW3: Schrödinger equation (set Sep 17, due Sep 24)

1. Write down the time-dependent Schrödinger equation (TDSE) and explain each term.
   (a) Why is the TDSE first order in time while Newton's law $f = m\ddot{x}$ is second order?
   (b) What does the Hamiltonian $\hat{H}$ represent physically? Show that the wavefunction $\Psi(x,t) = A\mathrm{e}^{-\mathrm{i}(\omega t - kx)}$ satisfies the TDSE. What is the physical significance of ω and k, $\hat{H}$, and how they are related to differential operators?
   (c) Explain the condition(s) required and show the derivation from the TDSE to reach the time-independent counterpart. What meant physically by *stationary state*?
   (d) Integrate the TDSE to derive how an initial wavefunction $\phi(t=0) = \phi_0$ evolves with time. For a wavefunction initially in state $\phi(t=0) = \mathrm{e}^{\mathrm{i}\pi/2}$ with energy E, what is the wavefunction at time $t = T$?

2. By taking the time derivative of the probability density $\rho = \Phi^*\Phi$ for the wavefunction $\Phi(z,t)$ and applying Schrödinger's equation, show that it satisfies
$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial z} = 0. \qquad (.22)$$
   Find the expression for $j(z,t)$. What is the physical significance of this equation? Evaluate $j(z)$ for the plane wave states $\psi = \mathrm{e}^{\pm \mathrm{i}kz}$ and how is this related to velocity?

3. A free particle moves on a circle with radius R obeying Schrödinger's equation with $V(x) = 0$ and periodic boundary conditions $\psi(x) = \psi(x + 2\pi R)$. Show that the momentum p_n is integer n quantized and derive an expression for the energy E_n.

4. Write down the Schrödinger equation for wavefunction $\psi(x)$ in the potential
$$V(x) = \begin{cases} 0, & |x| < \frac{a}{2}, \\ \infty & \text{otherwise.} \end{cases} \qquad (.23)$$
   (a) What is the boundary condition at $x = \pm a/2$? Show that the energy eigenvalues E_n are integer n quantized and derive the wavefunctions $\psi_n(x)$ for $n = 1,2,3$.
   (b) Determine the normalization constant for the wavefunctions in the ground $\psi_1(x)$, first $\psi_2(x)$ and second $\psi_3(x)$ excited states.
   (c) Sketch the forms of the wavefunction $\psi_n(x)$ and probability distributions $|\psi_n(x)|^2$.
   (d) Show that the wavefunctions $\psi_1(x)$ and $\psi_2(x)$ are orthogonal
$$\int_{-a/2}^{a/2} \psi_1(x)\psi_2(x) = 0. \qquad (.24)$$

HW4: Bound states and wavepacket (set Sep 24, due Oct 1)

1. Consider the finite-well potential in section 6.4 with potential

$$V(x) = \begin{cases} -V_0, & |x| < a, \\ 0, & |x| > a. \end{cases} \tag{.25}$$

   (a) Explain the concept of parity, why the Hamiltonian with the above $V(x)$ has parity symmetry, and what this implies for wavefunction $\psi(x)$ solutions.

   (b) Consider bound states such that $-V_0 < E < 0$. Write down the time-independent Schrödinger equation for the three regions: $x < -a, -a < x < a, x > a$.

   (c) What functions (e.g. exponential, trigonometric functions) do the odd-parity wavefunction solutions take in the three regions $x < -a, -a < x < a, x > a$? What expressions do the kinetic energy take in each region? Sketch the even and odd parity wavefunctions with the lowest energy states in relation to the potential (.25).

   (d) Show the steps to derive the condition that odd-parity wavefunctions must satisfy:

$$k\cot(ka) = -\kappa. \tag{.26}$$

   State the expressions for k and κ. Find the quadratic relation between k, κ, V_0 by eliminating E from Schrödinger's equation. Sketch this quadratic relation and $k\cot(ka) = -\kappa$, then explain qualitatively how to find the solutions graphically.

2. Consider a particle wavefunction $\psi(x)$ taking the form of a Gaussian wavepacket

$$\psi(x) = \frac{1}{(2\pi\sigma)^{1/4}} \exp\left(-\frac{x^2}{4\sigma^2}\right). \tag{.27}$$

   (a) What is the wavefunction normalization condition? Show that this wavefunction is normalized with the help of the standard Gaussian integral identity

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-(a^2x^2+bx)} = \frac{\sqrt{\pi}}{a} \mathrm{e}^{b^2/4a^2}. \tag{.28}$$

   (b) Explain qualitatively how the Fourier transform from position $\psi(x)$ to momentum $\psi(p)$ space, $\psi(p) = \frac{1}{\sqrt{h}} \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px/\hbar} \psi(x)$, is analogous to a discrete Fourier series. What is the physical interpretation in terms of quantum superposition?

   (c) Perform the Fourier transform integral to show that the wavefunction $\psi(p)$ and its probability distribution $|\psi(p)|^2$ are also Gaussians.

   (d) How are the standard deviations of the wavepacket in position and momentum spaces related mathematically? Comment on its physical significance.

HW5: Scattering and Dirac notation (set Oct 1, due Oct 8)

1. Consider the scattering problem through a thick-wall potential

$$V(x) = \begin{cases} V_0, & 0 < x < a, \\ 0, & \text{otherwise.} \end{cases} \tag{.29}$$

   (a) Write down the Schrödinger equation and the functional forms of the wavefunctions in the three regions $x < 0, 0 < x < a, x > a$ for (i) $E < V_0$ and (ii) $E > V_0$.

   (b) For $E > V_0$, apply appropriate boundary conditions and the definitions of probability currents j to derive the transmittivity $T = |j_{\text{transmitted}}/j_{\text{incident}}|$

$$T = \frac{1}{1 + \frac{1}{4}\left(\frac{k}{K} - \frac{K}{k}\right)^2 \sin^2(Ka)}. \tag{.30}$$

   (c) What is the condition on K for $T = 1$ and thus reflectivity $R = 0$ to vanish?

2. A vector space V has position vectors $\mathbf{r} = (x, y, z)^{\mathrm{T}}$. How do we write $\mathbf{r}$ in terms of basis vectors. How do we write a scalar product with another vector $\mathbf{r} \cdot \mathbf{r}'$ in Dirac notation? What is meant by a Hilbert space and adjoint space?

3. Given a state $|\psi\rangle = \mathrm{e}^{\mathrm{i}\pi/6}|a\rangle + (\mathrm{i}/2)|b\rangle$, what is its dual $\langle\psi|$?

4. How do we expand a state $|\psi\rangle$ with basis states $|n\rangle$ and coefficients a_n? In quantum mechanics, what property must $\langle\psi|\psi\rangle$ satisfy and what is the significance of a_n?

5. How do we mathematically define the inner product $\langle\psi|\phi\rangle$ for finite-dimensional vectors and infinite-dimensional functions?

6. Like the position representation, we can write a state $|\psi\rangle$ in momentum representation

$$|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}p\, \psi(p)|p\rangle. \tag{.31}$$

   (a) Explain each object $|\psi\rangle$, $\int_{-\infty}^{\infty} \mathrm{d}p$, $\psi(p)$ and $|p\rangle$ in the language of probability amplitudes and basis vectors of a Hilbert space.

   (b) How do we extract the wavefunction in momentum space $\psi(p)$ from state $|\psi\rangle$?

   (c) How do we write the orthogonality condition for the momentum basis states $|p_1\rangle$ and $|p_2\rangle$?

HW6: Operators and measurement (set Oct 8, due Oct 15)

1. What is the definition of a *Hermitian operator* and why is this important for physics? If $\hat{X}$ and $\hat{P}$ are Hermitian operators, show that $\mathrm{i}[\hat{X},\hat{P}]$ is also Hermitian.

2. What is the condition for an operator $\hat{Q}$ acting on wavefunctions $\psi(x), \phi(x)$ to be Hermitian in Dirac and integral notation? Why do $\psi(x)$ and $\phi(x)$ vanish as $x \to \infty$? Prove via integration by parts that the momentum operator $\hat{p} = -\mathrm{i}\hbar\frac{\partial}{\partial x}$ is Hermitian.

3. Consider the state $|\psi\rangle = a|A\rangle + b|B\rangle$.

   (a) What is the amplitude $\langle A|\psi\rangle$ and explain your answer? Upon measurement, what is the probability that the state is in $|A\rangle$ and $|B\rangle$?

   (b) What is the probability of measuring state $|A\rangle$ if (i) $a = \mathrm{e}^{\mathrm{i}\pi}$, (ii) $b = \mathrm{i}/2$?

4. Given the adjoint of an operator product is $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$, show that $(\hat{A}\hat{B}\hat{C})^\dagger = \hat{C}^\dagger\hat{B}^\dagger\hat{A}^\dagger$. By explicitly writing out the left and right hand sides, prove the following identities

$$[\hat{A},\hat{B}+\hat{C}] = [\hat{A},\hat{B}] + [\hat{A},\hat{C}], \tag{.32}$$

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}. \tag{.33}$$

5. Consider two observables of Hermitian operators $\hat{A}$ and $\hat{B}$ with eigenvalue equations

$$\hat{A}|a_n\rangle = a_n|a_n\rangle, \quad \hat{B}|b_n\rangle = b_n|b_n\rangle. \tag{.34}$$

   (a) Given $\hat{A}$ is Hermitian, show that its (i) eigenvalues must be real, (ii) eigenstates are mutually orthogonal. If the $[\hat{A},\hat{B}] = 0$ commute, show that they are simultaneous eigenstates of both $\hat{A}$ and $\hat{B}$. Why are these results important for physics?

   (b) Express the state $|\psi\rangle$ as a expansion of eigenstates $|a_n\rangle$ weighted by amplitudes $\gamma_n = \langle a_n|\psi\rangle$. Using this, derive the expectation value of $\hat{A}$ when acting on $|\psi\rangle$.

   (c) Consider $[\hat{A},\hat{B}] \neq 0$ on state $|\psi\rangle$. Let us measure one observable first, then the other immediately after without resetting the system. Show that the probability of measuring $\hat{A}$ first then $\hat{B}$ is $P = |\langle b_n|a_n\rangle|^2|\langle a_n|\psi\rangle|^2$. Find the probability of measuring $\hat{B}$ first then $\hat{A}$ and show that it differs. Discuss the physical significance.

6. By writing out the differential operators and acting on a general wavefunction $\psi(\mathbf{x})$, prove the canonical commutation relation for any number of dimensions $i, j = 1, 2, \ldots, N$

$$[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}, \tag{.35}$$

   where the Kronecker delta is $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. What is the physical significance of this commutator for $i = j$ and $i \neq j$? Why does $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$?

HW7: Harmonic oscillator and conservation (set Oct 22, due Oct 29)

1. The harmonic oscillator Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ and ladder operators are

$$\hat{A} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + \mathrm{i}\hat{p}), \quad \hat{A}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - \mathrm{i}\hat{p}). \tag{.36}$$

   (a) Taking the product $\hat{A}\hat{A}^\dagger$, show that the $\hat{H}$ can be written as $\hat{H} = \left(\hat{A}\hat{A}^\dagger - \frac{1}{2}\right)\hbar\omega$.

   (b) Show that the commutator $\hat{H}$ with the lowering operator is $[\hat{H}, \hat{A}] = -\hbar\omega\hat{A}$.

   (c) Use the previous result to show that $\hat{A}$ acting on $|E_n\rangle$ yields the eigenvalue equation $\hat{H}(\hat{A}|E_n\rangle) = (E_n - \hbar\omega)(\hat{A}|E_n\rangle)$. Comment on the physical significance.

   (d) Consider the operator $\hat{N} = \hat{A}^\dagger\hat{A}$ with $\hat{N}|n\rangle = n|n\rangle$ so we can label energy states by integers n e.g. $|n\rangle = |E_n\rangle, |n \pm 1\rangle = |E_n \pm \hbar\omega\rangle$. By taking the modulus squares $|\hat{A}|n\rangle|^2 = \langle n|\hat{A}^\dagger\hat{A}|n\rangle$ and $[\hat{A}, \hat{A}^\dagger] = 1$, show that $\alpha = \sqrt{n}$ in the eigenvalue equation

$$\hat{A}|n\rangle = \alpha|n-1\rangle. \tag{.37}$$

   (e) Derive the ground state wavefunction $\langle x|E_0\rangle$ by solving a differential equation, and show that the excited state $\psi_n(x) = \langle x|E_n\rangle$ is $\langle x|E_1\rangle = Ax\mathrm{e}^{-x^2/4\ell^2}$, What is the constant A? Explain why the wavefunctions have well-defined parity.

2. For two non-commuting observables $[\hat{A}, \hat{B}] \neq 0$, what inequality must the product of their uncertainties satisfy? What is the physical significance of this? What is this inequality when considering position $\hat{x}$ and momentum $\hat{p}$? Explain physically why the zero-point energy of the harmonic oscillator is non-zero.

3. Consider an operator $\hat{Q}$ that is time dependent $\partial\hat{Q}/\partial t \neq 0$.

   (a) Taking the time derivative of its expectation value $\langle\hat{Q}\rangle = \langle\psi|\hat{Q}|\psi\rangle$ and using the Schrödinger equation, present the derivation for Ehrenfest's theorem

$$\frac{\mathrm{d}\langle\hat{Q}\rangle}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\langle\psi|[\hat{H}, \hat{Q}]|\psi\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle. \tag{.38}$$

   (b) If an operator $\hat{A}$ is time independent $\partial\hat{A}/\partial t = 0$ and commutes with the Hamiltonian, what is the physical significance of this equation?

   (c) If the system is in an energy eigenstate $|E\rangle$ such that $\hat{H}|E\rangle = E|E\rangle$, show that $\langle\hat{A}\rangle$ is conserved even if it does not commute with the Hamiltonian $[\hat{A}, \hat{H}] \neq 0$.

   (d) For $\hat{Q} = \hat{H}$, what physically is conserved?

HW8: Angular momentum (set Oct 29, due Nov 5)

1. The projection of the orbital angular momentum operator along the x axis is

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y. \tag{.39}$$

   (a) By using the operator identity $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$ and canonical commutation relation $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$, show that $\hat{L}_x$ is Hermitian. Why does this imply that $\hat{L}^2$ is Hermitian?

   (b) By writing $[\hat{L}_x, \hat{L}_y]$ as $[\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$, prove the commutator

$$[\hat{L}_x, \hat{L}_y] = \mathrm{i}\hbar\hat{L}_z. \tag{.40}$$

   Discuss its physical significance. Hint: the $\mathrm{i}\hbar$ arises from the commutator $[\hat{z}, \hat{p}_z]$.

   (c) By explicitly expanding the left- and right-hand side, prove the commutator identity $[\hat{A}^2, \hat{B}] = \hat{A}[\hat{A}, \hat{B}] + [\hat{A}, \hat{B}]\hat{A}$. By writing out the definition of $\hat{L}^2$, use this to prove that $[\hat{L}^2, \hat{L}_y] = 0$. What is the physical significance of this?

2. Write down the definition of the angular momentum lowering operator $\hat{L}_-$.

   (a) Prove that $\hat{L}_-$ satisfies the commutator $[\hat{L}_z, \hat{L}_-] = -\hbar\hat{L}_-$.

   (b) Why does the lowering operator commute with the $\hat{L}^2$ i.e. $[\hat{L}^2, \hat{L}_-] = 0$?

   (c) Show that $|\psi\rangle = \hat{L}_-|\lambda, m\rangle$ is an eigenstate of $\hat{L}_z$ and find its eigenvalue.

   (d) Write down the eigenvalue equations for $\hat{L}^2$ and $\hat{L}_z$. Show that

$$\hat{L}_-^\dagger\hat{L}_- = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z. \tag{.41}$$

   Use this to show the eigenvalue equation for the lowering operator is

$$\hat{L}_-|\lambda, m\rangle = \alpha_-|\lambda, m-1\rangle. \tag{.42}$$

   Find the expression for α_-. Comment on the physical significance of this result.

3. Consider spherical coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$. Apply the chain rule to show that the $\hat{L}_z$ operator in spherical coordinates is

$$\hat{L}_z = -\mathrm{i}\hbar\frac{\partial}{\partial\phi}. \tag{.43}$$

HW9: Spherical Schrödinger problem (set Nov 5, due Nov 12)

1. The total angular momentum operator in spherical coordinates (r, θ, ϕ) is

$$\hat{L}^2 = -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]. \tag{.44}$$

   Why do $\hat{L}^2$, $\hat{L}_z$, and the Hamiltonian $\hat{H}$ commute for central potentials $V = V(r)$? Explain the physical significance.

2. Write down the eigenvalue equations for $\hat{L}^2$ and $\hat{L}_z$. Show that the following spherical harmonics are (i) eigenstates of $\hat{L}^2$ and $\hat{L}_z$, (ii) orthogonal by integration:

$$Y_1^0 = \sqrt{\tfrac{3}{4\pi}} \cos\theta, \quad Y_1^1 = -\sqrt{\tfrac{3}{8\pi}} \mathrm{e}^{+\mathrm{i}\phi} \sin\theta, \quad Y_1^{-1} = \sqrt{\tfrac{3}{8\pi}} \mathrm{e}^{-\mathrm{i}\phi} \sin\theta. \tag{.45}$$

3. The time-independent Schrödinger equation in spherical coordinates is

$$\left[-\frac{\hbar^2}{2m_e} \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}}{\mathrm{d}r} \right) + \frac{\hat{L}^2}{2m_e r^2} + V(r) \right] \psi(r, \theta, \phi) = E\psi(r, \theta, \phi). \tag{.46}$$

   (a) Comment on the physical significance of each term in this equation. Why can we factorize the wavefunction into a product of radial and angular functions $\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$? Why can we replace the $\hat{L}^2$ operator with $\ell(\ell+1)\hbar^2$?

   (b) Assuming the Coulomb potential $V(r) = -e^2/4\pi\varepsilon r$, explain the concept of an effective potential $V_{\text{eff}}(r)$ and what is the expression for this? What is the physical significance of the relative sign in front of the $\hat{L}^2$ and Coulomb terms?

   (c) Sketch the Coulomb potential, $\ell(\ell+1)\hbar^2/2m_e r^2$ piece, and effective $V_{\text{eff}}(r)$ for cases when the electron has (i) no angular momentum $\ell = 0$ and (ii) one unit of angular momentum $\ell = 1$. In the $\ell = 1$ case, explain the physical behaviour near and far from $r = 0$.

   (d) Change variables to $u(r) = rR(r)$ and show the radial equation becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2} u + \frac{2}{r_B r} u = k^2 u. \tag{.47}$$

   What are the expressions for k and r_B? Explain their physical interpretation.

   (e) Show the derivations for the functional form of the radial wavefunction $R(r)$ in the asymptotic limits: (i) near $r/r_B \to 0$ and far (ii) $r/r_B \to \infty$ from the nucleus. You do not need to normalize the wavefunction.

HW10: Spin half and hydrogen (set Nov 24, due Dec 3)

1. The spin operator is $\hat{\mathbf{S}} = \hat{\boldsymbol{\sigma}}/2$. Write down the three 2×2 Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$.

   (a) Describe, in terms of operators and eigenvalues, the physical and mathematical similarities and differences between electron spin and orbital angular momentum.

   (b) By explicitly matrix multiplication, show that: (i) commutator $[\hat{S}_y, \hat{S}_z] = \mathrm{i}\hbar\hat{S}_x$, and (ii) anticommutators $\{\hat{\sigma}_x, \hat{\sigma}_y\} = 0$ and $\{\hat{\sigma}_x, \hat{\sigma}_x\} = 2\hat{I}$, where $\hat{I}$ is the identity.

2. A system of two Stern-Gerlach (SG) machines sees the first aligned along the z-axis then a second SG device aligned along $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}$:

$$\xrightarrow[\text{spin-half particles}]{} \boxed{\mathrm{SG}_A} \xrightarrow[\text{transmits } |\uparrow\rangle \text{ along } \mathbf{z}]{} \boxed{\mathrm{SG}_B} \xrightarrow[\text{transmits } |\lambda_+\rangle \text{ along } \mathbf{n}]{} \qquad (.48)$$

   (a) Electrons in state $|\chi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ enter SG_A. Show that the expectation value of the force on the electrons is $\langle \hat{f}_z \rangle = (|\alpha|^2 - |\beta|^2)\mu_B \frac{\partial B_z}{\partial z}$, where $\mu_B = e\hbar/2m_e$.

   (b) An electron enters SG_B in a spin-up state $|\uparrow\rangle$. By solving the eigenvalue problem $\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}|\lambda_-, \mathbf{n}\rangle = \lambda_-|\lambda_-, \mathbf{n}\rangle$, show that the probability of measuring the spin-down state $|\lambda_-, \mathbf{n}\rangle$ after exiting SG_B is $P(\lambda_-, \mathbf{n}|\uparrow) = \sin^2\left(\frac{\theta}{2}\right)$.

3. Consider a system of two identical electrons.

   (a) Explain the concept of identical particles and the Pauli exclusion principle.

   (b) By applying the combined spin raising operator $\hat{S}_{+,12} = \hat{S}_{+,1} + \hat{S}_{+,2}$ onto the $|1,-1\rangle = |\downarrow\rangle_1|\downarrow\rangle_2$ state, show that this gives $|1,0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle_1|\uparrow\rangle_2 + |\uparrow\rangle_1|\downarrow\rangle_2)$. Explain why the $|1,0\rangle$ state is considered entangled whereas $|1,-1\rangle$ is separable?

4. Describe the three quantum numbers and their physical interpretation that characterize the states of hydrogen gross structure. How are their allowed values?

   (a) Express the energies E_n in terms of the fine-structure constant α and electron mass m_e. What is the value of E_1 in eV and Joules? How does this change if we use the reduced mass $\mu = m_e m_p/(m_e + m_p)$ instead of a stationary proton?

   (b) What states can electrons occupy in the ground, first and second excited energies? What does *degeneracy* mean in this context? Explain your answers by counting states with the hydrogen quantum numbers and concept of electron (sub)shells.

   (c) What is meant by a hydrogenic atom? Write the expressions for the energy formula, ground and first excited state wavefunctions in terms of atomic number Z. How does the most probable radial electron position scale with Z?